Arithmetic Strength

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Abstract

Strength is an invariant of homogeneous multivariable polynomials, roughly measuring the "factorization complexity" of the polynomial. **Collective strength** is an extension of strength to collections of polynomials. These invariants have been independently reinvented several times in disparate fields, most recently in 2016 by Ananyan and Hochster in their proof of Stillman's Conjecture in commutative algebra. Although there has been much work on strength, with many recent results motivated by its appearance in Ananyan-Hochster, most of the study of strength has occurred over algebraically closed fields. However, we find that having low strength is a generalization of a polynomial admitting a rational solution, which opens up new avenues to explore. Motivated by this, we explore a range of enumerative and statistical questions about strength in arithmetic settings — over $\mathbb{F}_p$, $\mathbb{Q}_p$, and $\mathbb{Q}$ — particularly focusing on the case of curves.

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Contents

Chapter 1

Introduction

The central focus of this thesis will be an invariant of a multivariable polynomial called **strength**. Throughout, we will usually tacitly assume our polynomials are homogeneous, meaning each term has the same total degree (that is, the same number of not-necessarily-distinct variables multiplied together). For example, in the variables x, y, z, the polynomial $x^3 + yz^2$ is homogeneous of degree 3, while the polynomial $x + y^2 + z^3$ is inhomogeneous.

One can think of strength as a measure of the factorization complexity of a polynomial. The simplest kinds of polynomials are reducible polynomials, polynomials f with a factorization $f = gh$, where g and h are nonconstant polynomials. The polynomial $x^2+xy+xz+yz$ factors as $(x+y)(x+z)$. The next simplest kind of polynomial would be one which can be written as a sum of two reducible polynomials. That is, a polynomial f which can be written as $f = g_1h_1+g_2h_2$, where again the g_i and h_i should be nonconstant polynomials. For example, the polynomial $x^2 + yz$ is irreducible, but it can clearly be written as $x \cdot x + y \cdot z$, a sum of two reducible polynomials. In general, the strength of a polynomial f, which we denote $\mathrm{str}(f)$, is the minimum number of addends needed so that f has a decomposition as a sum of reducible polynomials:

$$f = \underbrace{g_1h_1 + \cdots + g_rh_r}_{\mathrm{str}(f)=\text{fewest needed}}.$$

We have two border cases: $f = 0$ and $\deg(f) = 1$. We extend our definition by declaring that $\mathrm{str}(0) = 0$, and that $\mathrm{str}(f) = \infty$ if f is linear. (Linear polynomials are infinitely complex under this metric: no matter how hard you try to decompose them, you will never succeed!)

We also speak of the strength of a collection of polynomials $f_1, f_2, \ldots, f_m$. The **collective strength** of the collection, which we will denote by $\mathrm{str}(f_1, \ldots, f_m)$, will be the least strength among polynomials of the form $a_1 f_1 + \cdots + a_m f_m$, where the a_i are constants not all equal to zero, and the sum is again a homogeneous polynomial. For example, if the f_i all have different degrees, then the only such sums have a single nonzero a_i, and so the collective strength of the f_i will simply be the minimum of $\mathrm{str}(f_i)$.

Example 1.1.

- Over the complex numbers $\mathbb{C}$, the polynomial $x^2 + y^2$ has strength 1, as it factors as $(x + iy)(x - iy)$. However, over the real numbers $\mathbb{R}$, it has strength 2, since this polynomial is irreducible over $\mathbb{R}$ but manifestly admits a decomposition $x \cdot x + y \cdot y$.
- Over $\mathbb{C}$, the polynomial $f = x^2 + y^2 + z^2$ is irreducible, so $\mathrm{str}(f) \geq 2$. It admits the decomposition $(x + iy)(x - iy) + z \cdot z$, and so $\mathrm{str}(f) = 2$.
- Still over $\mathbb{C}$, the polynomial $g = 2xy + z^2$ is also strength 2, as it differs from f by a linear change of variables, and strength is invariant under linear change of variables. However, the collective strength $\mathrm{str}_{\mathbb{C}}(f, g)$ is 1, since $f - g = x^2 - 2xy + y^2 = (x - y)^2$.

Strength and collective strength have an interesting history, having been independently reinvented more than once in different areas of math. They first appeared in analytic number theory, introduced by Schmidt in 1985 under the name "h-rank" [34] in the context of applying the Hardy-Littlewood circle method to estimate the number of rational points on projective varieties when the number of variables is large. Strength was then used by Green and Tao in 2009 in work on higher-order Fourier analysis over finite fields [24]. More recently, it was used by Ananyan and Hochster [2] in 2016, who gave it the name "strength", in their proof of Stillman's Conjecture, a conjecture in commutative algebra concerning whether the projective dimension of a homogeneous ideal in a polynomial ring can be bounded in terms of the degrees of its generators, independent of the number of variables. Strength is also related to tensor decompositions, since it is a symmetrized version of what tensor researchers call “partition rank”, first defined by Naslund in 2017 [31].

Since the work of Ananyan and Hochster, there has been interest among commutative algebraists and algebraic geometers to investigate its behavior. As we will discuss more later, questions about strength fit in to classical questions in algebraic geometry about large linear subvarieties of projective varieties, or more generally large complete intersection subvarieties. That is, in turn, connected to studying certain secant varieties, another area with a long history. However, most of the work on strength is in the setting of strength over $\mathbb{C}$. We wish to explore some of the properties of strength in over other fields, in particular in arithmetic settings familiar to number theorists: over finite fields, over nonarchimedean local fields, and over the rational numbers and other number fields.

We are motivated in part by the following analogy between strength and the study of rational points. Over any field k, if $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree at least 2, then the strength of f is at most $n+1$, since f may always be expressed as $x_0h_0 + \cdots + x_nh_n$. Geometrically, the polynomial f defines a projective variety $V(f) \subset \mathbb{P}^n$ corresponding to the solutions to $f = 0$. We say f has a k-point if $f = 0$ admits a solution $f(a_0, \ldots, a_n) = 0$ where the a_i are in k. If f has a k-point, then is contained in the ideal $(\ell_1, \ldots, \ell_n)$, where ℓ_i is the linear polynomial $a_0x_i - a_ix_0$. So, there is an equality $f = \ell_1h_1 + \cdots + \ell_nh_n$ for some non-constant polynomials h_i. That is, if $f = 0$ has a k-point, then $\mathrm{str}(f) \leq n$. Thus, “f has less than maximal strength” can be viewed as a generalization of “f has a k-point”. (More generally, if $V(f)$ contains a k-rational linear subvariety of codimension r, then $\mathrm{str}(f) \leq r$.) Investigating when polynomials admit rational points, and when they do how many there are and how to find them, constitute central questions in arithmetic geometry. Thus, questions which have been investigated in the context of rational points may be reformulated into new questions in the context of strength.

There are several reasons an arithmetic study of strength should be of interest:

- Strength or "h-rank" continues to play a role in analytic number theory, with recent examples including [29, 33]. Collections of polynomials with high strength tend to be amenable to the circle method, so characterizing high-strength loci can produce theorems about rational points on varieties when the degree is low relative to the ambient dimension.

- There is a large literature investigating how often a "random" polynomial is reducible in arithmetic situations [5, 6, 14, 23]. Generalizing this program, one can ask how often a random polynomial has strength $\leq r$, or how often a random collection $f_1, \ldots, f_m$ has collective strength $\leq r$.

- The geometric question of whether $V = V(f_1, \ldots, f_m)$ contains a large linear subspaces is related to other questions about rational points and whether the variety V is rational. Examples of other work investigating large linear subvarieties with applications to rational points include [7, 26].

Many of the results and applications involving strength take place in the setting where the number of variables is much larger than the degree of the forms being considered. Our work is in a different regime: we focus on the case of curves and vary the degree and number of polynomials considered. Our main result is the following theorem about the collective strength of two quadric curves over various fields.

Theorem 3.3. Let k be a number field, a finite field, or a nonarchimedean local field.

(a) If k is a number field, then consider the set of all pairs of ternary quadratic polynomials (f_1, f_2) with coefficients of height at most B. Let $N(B)$ be the total number of such pairs, and let $N_{\mathrm{str}=1}(B)$ be the number with strength 1. Then $N_{\mathrm{str}=1}(B)/N(B) = O((B^{[k:\mathbb{Q}]/2}\log B)^{-1})$.

(b) If $k = \mathbb{F}_q$ if a finite field, then the number of pencils of conics in the strength 2 locus of $\mathrm{Hilb}_4(\mathbb{P}^2)$ is $\frac{7}{12} \cdot \#\,\mathrm{PGL}_3(\mathbb{F}_q)$.

(c) If k is a nonarchimedean local field, let $\mathcal{O}_k$ be the ring of integers, $\mathfrak{p}$ the maximal ideal, $\mathcal{O}_k/\mathfrak{p} = \mathbb{F}_q$ the residue field, and μ the Haar measure normalized so that $\mu(\mathcal{O}_k) = 1$. The measure of the set of pencils (f_1, f_2) with $f_i \in \mathcal{O}_k$ and $\operatorname{str}(f_1, f_2) = 2$ differs from $\frac{7}{12} \cdot \frac{\# \mathrm{PGL}_3(\mathbb{F}_q)}{\# \mathbb{P}^8(\mathbb{F}_q)}$ by $O(q^{-2})$.

The structure of this thesis is as follows. In section 2.1 we lay out all the definitions and notation which will be needed for our discussion of strength. In section 2.2, we survey some prior work on strength, including what preexisting work there is on strength in the arithmetic setting. In section 2.3, we will propose some open questions about the arithmetic behavior of strength, which guide our study in the following chapter. Our main results on the arithmetic strength of curves are contained in Chapter 3. In section 3.1, we undertake a census of the arithmetic strength of curves of low degree, recasting many existing results in the language of strength, and providing new observations and computations in the remaining cases. Finally, section 3.2 contains our results on the arithmetic strength of a pair of ternary quadratic polynomials.

Chapter 2

Background

2.1 Definitions and Notation

Fix a field k and an integer $n \geq 1$. Let $S = k[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables. Let S_d be the vector space of homogeneous forms of degree d, and let $\mathcal{P}_d = \mathbb{P}S_d$ be its projectivization. Note that $\dim \mathcal{P}_d = \binom{n+d}{d} - 1$. By a k-variety, we mean a reduced, separated scheme over k.

Definition 2.1.

- The **strength** of a homogeneous $f \in S$, denoted $\mathrm{str}_k(f)$ or $\mathrm{str}(f)$ when the field is understood, is the least integer r so that f can be written as a sum of r reducible polynomials, $f = g_1 h_1 + \cdots + g_r h_r$. We define $\mathrm{str}(0) = 0$ and $\mathrm{str}(f) = \infty$ if $\deg(f) = 1$.

- If $f_1, \dots, f_m$ are homogeneous polynomials of possibly different degrees, their **collective strength**, $\mathrm{str}_k(f_1, \dots, f_m)$, is the minimum strength of a homogeneous k-linear combination of the f_i.

- We call a particular expression like $f = g_1 h_1 + \cdots + g_r h_r$ a **strength** r **decomposition of** f. We could therefore say that $\mathrm{str}(f)$ is the least r such that f admits a strength r decomposition.

- Suppose $f = g_1h_1 + \cdots + g_rh_r$ with $\deg(g_i) \leq \deg(h_i)$ and $\deg(g_i) \leq \deg(g_{i+1})$. Then we say the **type** (or sometimes **strength type** for clarity) of this strength decomposition is the nondecreasing sequence $(\deg(g_1), \deg(g_2), \ldots, \deg(g_r))$.

We wish to find a geometric description of the locus in $\mathcal{P}_d$ consisting of polynomials of strength r. We begin with the subvariety of reducible forms. Multiplication of polynomials induces regular maps

$$\begin{aligned} \mathcal{P}_a \times \mathcal{P}_{d-a} &\to \mathcal{P}_d \\ (g, h) &\mapsto gh. \end{aligned}$$

These maps are proper over k since each $\mathcal{P}_d$ is separated and proper. Thus, the image of this map is a closed subvariety of $\mathcal{P}_d$ consisting of those polynomials f of degree d admitting a factorization $f = gh$ with $\deg g = a$. We denote this variety by X_a. The union $Y_d = \bigcup_{a \leq d/2} X_a$ is the variety of all reducible polynomials of degree d.

To talk about strength, we want not just the variety of reducible forms, but the variety of sums of reducible forms. The construction we need here is the *secant variety* of the variety of reducible forms

Definition 2.2. Given two distinct points x, y in a projective space, the **join** (or **linear span**) of x and y, denoted $\langle x, y\rangle$, is the unique line passing through x and y. Given two subvarieties of a common projective space, X and Y, we will also speak of the join of X and Y, denoted by $\langle X, Y\rangle$. The join of X and Y is the Zariski closure of all lines $\langle x, y\rangle$ where $x \in X, y \in Y$ are distinct:

$$\langle X, Y\rangle = \overline{\bigcup_{\substack{x \in X \\ y \in Y \\ x \neq y}} \langle x, y\rangle}.$$

For more than two points, we define $\langle x_1, \ldots, x_k\rangle$ to be the unique $k-1$ plane containing the x_i. Equivalently, $\langle x_1, x_2, x_3\rangle = \langle x_1, \langle x_2, x_3\rangle\rangle$, and so on. We extend this in the obvious way to more than one variety. Finally, in the case $X = Y$, we will define the r**th secant**

variety of X, $\sigma_r(X)$, to be

$$\langle \underbrace{X, \ldots, X}_{r} \rangle.$$

Concretely, the rth secant variety is (the Zariski closure of) the union of all $r-1$-planes meeting X at r distinct points.

Remark 2.3. The reason we need to take the closure in Definition 2.2 is that the union of the joins is in general not closed: there can be lines which are limits of joining lines, but which are not themselves joining lines, e.g. tangent lines. For a simple example, consider $\mathbb{P}^2$ with projective coordinates x, y, z. Let $X = V(yz - x^2)$ the projective closure of the parabola $y = x^2$, and $Y = V(x, y)$ the origin. Then $\bigcup_{P \in X \setminus Y} \langle P, Y \rangle$ is the constructible set $\mathbb{P}^2 \setminus (V(y) \setminus V(x, y))$.

Thus, we have that the locus of degree d polynomials of strength $\leq r$ is dense in $\sigma_r(Y_d)$, the rth secant variety of the variety of reducible forms. In view of Remark 2.3, it is natural to wonder whether the locus of strength of forms of strength $\leq r$ is already Zariski closed, so that it is exactly the rth secant variety of Y_d. The answer is no, see Theorem 2.5 below. Some authors accordingly define a notion of **border strength**, where f has border strength r if it is contained in $\sigma_r(Y_d)$. We will not use this notion.

Given a variety $X \subset \mathbb{P}^n$, there is a dominant rational map $X^r \times \mathbb{P}^{r-1} \dashrightarrow \sigma_r(X)$ which sends $(x_1, \ldots, x_r) \times (t_1, \ldots, t_r)$ to $t_1 x_1 + \ldots t_r x_r \in \sigma_r(X)$. We see that the locus of forms of strength $\leq r$ is defined to be the image of this map when $X = Y_d$. A subtlety arises when working over k non-algebraically closed: a k-point of the image may not be the image of a k-point. Concretely, a form may be strength $\leq r$ after passing to the algebraic closure $\overline{k}$, but it may not have a rational strength r decomposition. We will see examples of this nature in chapter 3.

In what comes later, we will occasionally make use of asymptotic notation. We write $f(x) = O(g(x))$ to mean that there are constants a, b so that for all x large enough, $f(x) \leq a g(x) + b$. We write $f(x) \sim g(x)$ to mean $\lim_{x \to \infty} f(x)/g(x) = 1$. We write $f(x) \asymp g(x)$ to mean there are constants c_1, c_2 so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for all x

sufficiently large.

We close this section by recalling some terminology relating to varieties and schemes. An affine scheme is a scheme of the form $\operatorname{Spec} A$ for some commutative ring A. A **closed point** of an affine scheme is the subscheme $\operatorname{Spec} A/\mathfrak{m}$ associated to a maximal ideal $\mathfrak{m} \subset A$. Since more general schemes are built from affine pieces, a closed point on an arbitrary scheme is a closed point in some affine patch. When k is algebraically closed, the Nullstellensatz tells us that the set of closed points in $\mathbb{A}^n$ corresponds exactly to k^n. When k is not algebraically closed, we may think of a closed point of $\mathbb{A}^n$ as being a Galois orbit of points in $\bar{k}^n$. The **degree** of a closed point is the dimension of the residue field $A/\mathfrak{m}$ as a k-vector space, which is the same as the number of points of $\bar{k}^n$ in the Galois orbit.

By a curve we mean a variety of dimension 1. A **divisor** on a curve is a formal sum $\sum_{P\in C} n_P[P]$ where each P is a closed point of the curve, and only finitely many n_P are nonzero. The degree of the divisor $\sum_{P\in C} n_P[P]$ is $\sum_{P\in C} n_P \deg(P)$.

2.2 Prior work on strength

Strength is completely understood over $\mathbb{C}$ and $\mathbb{R}$ in the case of a single polynomial of degree 2. For a quadratic form in $n+1$ variables, after diagonalization it looks like $x_0^2 \pm x_1^2 \pm \cdots \pm x_R^2$, where R is the rank of the quadratic form. We can pair up a $+x_j^2$ with a $-x_\ell^2$ to factor as $(x_j + x_\ell)(x_i - x_\ell)$. Over $\mathbb{C}$, by absorbing a factor of i, we can freely switch the signs, so we are always able to obtain a strength $\lceil R/2 \rceil$ decomposition $(x_0+x_1)(x_0-x_1)+\ldots$, where the last term is $(x_{R-1}+x_R)(x_{R-1}-x_R)$ if R is even and x_R^2 if R is odd. Over $\mathbb{R}$, we will always have a number of leftover terms equal to the absolute value of the signature of the form, and these we will not be able to factor any better than as x_j^2. So, over $\mathbb{R}$ we obtain a strength $\lceil (R+\mathrm{sig})/2 \rceil$ decomposition. These constructions are indeed optimal. Over $\mathbb{C}$, this follows from the following fact, although over $\mathbb{R}$ one must do more work.

Lemma 2.4. Let $f \in k[x_0,\ldots,x_n]$, and let Σ_f be the singular locus of $V(f)$. Then $2\operatorname{str}(f) \geq \operatorname{codim}_{\mathbb{P}^n} \Sigma_f$.

Proof. Suppose f has a strength r decomposition $f = g_1h_1 + \cdots + g_rh_r$. Then $\frac{\partial f}{\partial x_i} = g_1\frac{\partial h_1}{\partial x_i} + h_1\frac{\partial g_1}{\partial x_i} + \ldots$. Since Σ_f is cut out by the equations $\frac{\partial f}{\partial x_i} = 0$, $\Sigma_f \supset V(g_1, h_1, \ldots, g_r, h_r)$. Hence, $\operatorname{codim}_{\mathbb{P}^n} \Sigma_f \leq \operatorname{codim}_{\mathbb{P}^n} V(g_1, h_1, \ldots, g_r, h_r) \leq 2r$, as desired.

Since Ananyan and Hochster resolved Stillman's Conjecture, there has been further work on strength and bounding projective dimensions of homogeneous ideals. Some examples include [19, 16, 18]. There have also been explicit bounds computed for various small degrees, for example [27], which also lists several other works of this kind.

Additionally, there has been work looking to characterize geometric properties of the the (border) strength $\leq r$ locus $\sigma_r(Y_d)$. A few natural questions in this vein are:

(1) Is the set of strength r forms Zariski closed, or is it a proper subset of $\sigma_r(Y_d)$ (c.f. Remark 2.3)?

(2) Does strength have a geometric interpretation?

(3) What is $\dim \sigma_r(Y_d)$?

(4) Given a strength type $(a_1, \ldots, a_r)$, the locus of forms admitting a strength decomposition of that type is dense in $\langle X_{a_1}, \ldots, X_{a_n} \rangle$. When maximal, these subvarieties of $\sigma_r(Y_d)$ form the irreducible components. A more refined version of question (3) is: what are the dimensions of these components?

(5) What does a "typical" strength r decomposition look like?

To question (1), Ballico, Bik, Oneto, and Ventura gave the following counterexample.

Theorem 2.5 ([4] Theorem 1.3)**.** *Let k be algebraically closed. For polynomials x, y, u, v of degree 1 and f, g, p, q of degree 2, the polynomial*

$$x^2f + y^2g + u^2p + v^2q$$

is always a limit of polynomials of strength ≤ 3. However, if the number of variables n is sufficiently large, then for a suitable choice of x, y, u, v, f, g, p, q the above polynomial has strength 4.

To question (2), we can offer an interpretation based on the following fact.

Lemma 2.6 (generalization of [3] Lemma 2.1)**.** Suppose that k is infinite, and fix a strength type $(a_1, \ldots, a_r)$. A general f admitting a strength decomposition $f = g_1h_1 + \cdots + g_rh_r$ of type $(a_1, \ldots, a_r)$ admits one where the g_i form a regular sequence. Consequently, $V(f)$ contains the complete intersection $V(g_1, \ldots, g_r)$.

Proof. The subvariety of $\mathcal{P}_{a_1} \times \cdots \times \mathcal{P}_{a_r}$ where $(g_1, \ldots, g_r)$ is a regular sequence is open and dense. Since k is infinite, the k-points of $\mathcal{P}_{a_1} \times \cdots \times \mathcal{P}_{a_r}$ are dense, and hence the set of regular sequences (g_i) defined over k are dense. Therefore, the set of strength decompositions $g_1h_1 + \cdots + g_rh_r$ defined over k and where the (g_i) form a regular sequence is dense in $\langle X_{a_1}, \ldots, X_{a_r} \rangle$.

Say a complete intersection X is *nontrivially contained* in $V(f)$ if $X \subset V(f)$ and X is a complete intersection of hypersurfaces of degree less than $\deg(f)$ (note that it is quite easy to find complete intersections where $V(f)$ is one of the intersectands). Then, we have found that the strength of a general f is the minimal $\operatorname{codim}_{\mathbb{P}^n}(X)$, where X is a complete intersection nontrivially contained in $V(f)$. This perspective will occasionally be useful to us later because we can show that f has low strength by finding a complete intersection contained in it and showing that containment is nontrivial.

Progress has been made on questions (3)-(5) as well. Tools from the study of secant varieties have played a large role in these investigations. If the rational map $X^r \times \mathbb{P}^{r-1} \dashrightarrow \sigma_r(X)$ had generically finite fibers, then we would know that $\dim \sigma_r(X) = \dim(X^r \times \mathbb{P}^{r-1}) = r \dim X + r - 1$. This quantity is referred to as the **expected dimension** of the secant variety $\operatorname{edim}(\sigma_r(X))$, and we always have $\dim \sigma_r(X) \leq \operatorname{edim}(\sigma_r(X))$. A secant variety for which $\dim \sigma_r(X) < \operatorname{edim}(\sigma_r(X))$ is called **defective**, and the difference between the two is called the **defect**. A key tool for measuring the defect of secant varieties is Terracini's lemma.

Theorem 2.7 (Terracini's Lemma, c.f. [8] Lemma 2.13, [1] Corollary 1.10)**.** Let X be a projective variety, and let t_xX be the embedded tangent space to X at x, i.e. the linear subvariety of $\mathbb{P}^n$ tangent to X at x.

i. $\langle t_{x_1}X, \ldots, t_{x_r}X\rangle \subset t_{\langle x_1,\ldots,x_r\rangle}\sigma_r(X)$ for x_i points of X.

ii. If k is algebraically closed of characteristic 0, then there is an open dense subset of $\sigma_r(X)$ where equality holds. In particular, it is enough for the x_i to be distinct.

As regards strength, one can check that the embedded tangent space to X_a at a general point $[g \cdot h]$ is $\mathbb{P}(g, h)_d$, the projectivization of the degree d part of the ideal generated by g and h. Thus, over an algebraically closed field of characteristic 0, one can use Terracini's lemma to show that the embedded tangent space to a general point of $\langle X_{a_1}, \ldots, X_{a_r}\rangle$ is identified with the degree d part of some ideal. Therefore, by using Hilbert series, one can hope to get a good grasp on the dimension of $\langle X_{a_1}, \ldots, X_{a_r}\rangle$.

This is exactly what is done in [15], who actually consider a somewhat more general circumstance where there are more than two factors in the factorization. They were able to prove that $\dim \sigma_r(Y_d) = \dim \sigma_r(X_1)$ holds for all $r \leq n/2$. This was later improved by [3], who showed that in fact $\dim \sigma_r(Y_d) = \dim \sigma_r(X_1)$ for all r, and indeed that all the other components of Y_d have lower dimension. This also tells us that in some sense a "typical" strength r decomposition is of type $(1, \ldots, 1)$. Note that $\dim \sigma_r(X_1)$ is known: it is $\binom{n+d}{d} - \binom{n-r+d}{d} + r(n - r + 1)$.

Another avenue of research about strength are generalizations of the the following "universality" property, originally a result of Kazhdan and Ziegler [28]. Given $f \in k[x_0, \ldots, x_n]$ and linear forms $\ell_0, \ldots, \ell_n \in k[y_0, \ldots, y_m]$, we say $f(\ell_0, \ldots, \ell_n)$ is a *coordinate transform* of f. Suppose P is a property that polynomials of degree d (in any number of variables) might have. Assume further that P is such that f has P if and only if every coordinate transform of f has P. Then, either every polynomial of degree d has P, or else there is an absolute bound r_0 (independent of the number of variables) so that "f has P" implies $\mathrm{str}(f) \leq r_0$. Here, one should think that the property P is an undesirable property that one wishes to avoid, for example that the codimension of the singular locus of $V(f)$ is at most 2. (A similar statement applies to collective strength.) There have been elaborations on this "universality" of strength in papers such as those of Bik, Draisma, and Eggermont [12] and Bik, Danelon, Draisma, and Eggermont [13].

This universality helps explain why strength has been rediscovered so many times, by researchers with rather different backgrounds and goals. The property P = "the polynomials $f_1, \ldots, f_m$ do not form a regular sequence" is one possible property satisfying the above conditions, and both Schmidt and Ananyan-Hochster use the fact that once the collective strength of $f_1, \ldots, f_m$ is sufficiently high, they must fail to have P, i.e. they must be a regular sequence.

There has been some recent work on strength in arithmetic situations. As we noted above, a form may be strength $\leq r$ after passing to the algebraic closure $\overline{k}$, but strength $> r$ over k. In [29], Lampert and Ziegler study how much strength can change when passing from k to its algebraic closure. From their results, they are immediately able to conclude that certain kinds of Diophantine equations have solutions, and estimate the number of those solutions. Analogous work for tensor decompositions was undertaken in [17].

2.3 Motivating questions about arithmetic strength

Our present results are modeled on research on "random Diophantine equations". The questions in this area typically take the following form. Given some model for generating a variety given integer parameters, let $N(B)$ be the number of different choices of the model parameters among integers with absolute value at most B. Let $N_{\text{loc}}(B)$ be the number of varieties among those which admit points "everywhere locally", that is which admit points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime p. Let $N_{\text{glob}}(B)$ be the number of varieties which admit $\mathbb{Q}$-points. One would like to know about the asymptotic behavior of $\frac{N_{\text{glob}}(B)}{N(B)}$, which we call the (asymptotic) proportion of varieties in the family which are globally soluble. Often, one attacks this by first answering:

(1) What is the asymptotic behavior of $\frac{N_{\text{loc}}(B)}{N(B)}$ as $B \to \infty$?

(2) Does the **Hasse principle** (also called a local-to-global principle) hold, i.e. if a variety of this kind has points everywhere locally, does that mean it has $\mathbb{Q}$-points?

> If a local-to-global principle does not hold universally, does it hold for some positive proportion of the varieties in question?

Often to answer the local question, one must pass first to answer a corresponding counting question over the finite field $\mathbb{F}_p$.

Below, we list several questions which motivate our study of arithmetic strength. We remark on how our present results fit into these broader questions, and hope to pursue these questions in more detail in the future.

- In [32], Poonen and Voloch showed that, for fixed d, a positive proportion of $f \in \mathbb{Z}[x_0, \ldots, x_n]$ of degree d have a point over every $\mathbb{Q}_p$ except when $d = 2$ and $n = 2$. For which r is it true that a positive proportion of $f \in \mathbb{Z}[x_0, \ldots, x_n]$ of degree d have $\mathrm{str}_{\mathbb{Q}_p}(f) \leq r$ for all p? How does this depend on n, d? One can ask a similar question about the collective strength of $f_1, \ldots, f_m$ of degrees $d_1, \ldots, d_m$. Our results for pairs of conics correspond to the case $n = 2, m = 2, d_1 = d_2 = 2$. A Poonen-Voloch-style result trivially holds in that case with $r = 2$, but our results imply that the corresponding statement for $r = 1$ is false.

- To what extent could there be a local-global principle for strength? For example, could it be true that for $f \in \mathbb{Q}[x_0, \ldots, x_n]$, $\mathrm{str}_{\mathbb{Q}}(f) = \max_p \mathrm{str}_{\mathbb{Q}_p}(f)$? Our results for pairs of conics imply that $\mathrm{str}_{\mathbb{Q}}(f_1, f_2) = \max_p \mathrm{str}_{\mathbb{Q}_p}(f_1, f_2)$ is true for a density 1 set of pairs of conics, but do not establish that it is true for every pair.

- If there is a local-global principle, we can ask for an approximation theorem. Can we find a strength r decomposition of f over $\mathbb{Q}$ arbitrarily close to finitely many fixed strength r decompositions over various completions?

- Is there a Chebotarev density theorem for strength? That is, for a given polynomial $f \in \mathbb{Q}[x_0, \ldots, x_n]$ with $\mathrm{str}_{\mathbb{Q}}(f) = r$, can we characterize what proportion of primes p will have $\mathrm{str}_{\mathbb{Q}_p}(f) = s$ for $s = 1, \ldots, r$?

- An important question embedded in all the questions above is: is there an analog of Hensel's lemma for strength? Given $f_1, \ldots, f_m \in \mathbb{Z}_p[x_0, \ldots, x_n]$, when is

> $\mathrm{str}_{\mathbb{F}_p}(f_i \mod p) = \mathrm{str}_{\mathbb{Q}_p}(f_i)$? As part of our analysis of pairs of conics, we analyze the lifting problem for strength 1 decompositions in detail. We find that the pencils whose p-adic strength is higher than the mod p strength are contained in a codimension 2 subvariety.

An answer to the last question is interesting in the geometric setting as well. It is known that the locus of f so that $\mathrm{str}(f) \leq r_0$ is not Zariski closed in general, and similar results hold for other tensor ranks. One might ask, given $f \in \mathbb{C}[x_0, \ldots, x_n]$ of strength r_0, if we deform f what can happen to its strength? This is the same as asking about the strength of polynomials $F \in \mathbb{C}[[t]][x_0, \ldots, x_n]$ with $F \equiv f \mod t$. A result of Bik, Draisma, and Eggermont implies that there is *some* bound $R(r_0)$ independent of n so that if $\mathrm{str}(f) = r_0$, then the strength of a deformation is at most $R(r_0)$.

Chapter 3

Arithmetic Strength of Curves

We will begin with some observations about the behavior of strength for plane curves of low degree. We will then proceed to our main results about the collective strength of pairs of conics. Throughout this chapter, all polynomials considered will be in the three variables x, y, z.

3.1 Curves of Low Degree

In each subsection below, we will record what is known, or what is expected when no definitive results are to be found, about strength of curves of degrees $d = 2, 3$, and 4. We will divide our remarks between counting the number of curves of a given strength over $\mathbb{F}_p$, calculating the p-adic density of curves of a given strength decomposition over $\mathbb{Q}_p$, and giving asymptotics for curves of a given strength decomposition over $\mathbb{Q}$ as the naive height of the coefficients grows. Unless otherwise noted, the stated results hold also over $\mathbb{F}_q$, finite extensions of $\mathbb{Q}_p$, and number fields.

3.1.1 $d = 2$

For curves of degree 2, the only possible factorizations involve a linear term. Thus, a curve of degree 2 has strength 1 if and only if it is irreducible, and strength 2 if and only if it has a rational point.

$\mathbb{F}_p$: A degree 2 curve always admits an $\mathbb{F}_p$-point. For smooth curves, this follows from the fact that the Brauer group of a finite field is trivial, while a reduced singular quadric curve is geometrically a union of two distinct lines, and the intersection of the two components must be a rational point. We can count explicitly that there are $\frac{(p^2+p+1)(p^2+p+2)}{2}$ reducible conics over $\mathbb{F}_p$, and thus there are $p^5 + \frac{1}{2}p^4 - p^2 - \frac{1}{2}p = \frac{p(2p+1)(p^3-1)}{2}$ conics of strength 2.

$\mathbb{Q}_p$: For a degree 2 curve over $\mathbb{Z}_p$, we may always change variables so that the quadratic polynomial defining the curve is of the form $a'x^2+b'y^2+c'z^2$. Dividing out by any common factors of p, we can assume that at least one of $a', b', c' \notin p\mathbb{Z}_p$. Without loss of generality, say $c' \notin p\mathbb{Z}_p$, then dividing out by $-c'$ yields a quadratic of the form $ax^2 + by^2 - z^2$. Finally, by absorbing factors of p^2 from a and b into x^2 and y^2, we may assume that a and b each individually are either 0 or have valuation at most 1. The case where $ab = 0$ happens with p-adic density 0, so we assume $ab \neq 0$. In this case, whether the quadratic has a $\mathbb{Q}_p$-point depends on whether the Hilbert symbol $(a, b)_p$ is equal to 1 or -1. By basic properties of Hilbert symbols, for p odd this happens for a set of (a, b) of density $\frac{p^2+p+1/2}{(p+1)^2}$. For $p = 2$, this happens with density $\frac{11}{18}$.

$\mathbb{Q}$: Over $\mathbb{Q}$, the Hasse principle holds, so a conic has a $\mathbb{Q}$-point if and only if it has points over $\mathbb{R}$ and each $\mathbb{Q}_p$. By a result of Serre, when considering quadratics in $\mathbb{Z}[x, y, z]$ with coefficients bounded by B, the number with points everywhere locally is $N_{\text{loc}}(B)/N(B) = O((\log B)^{-1/2})$ [35]. That is to say, as $B \to \infty$, a random quadratic has a rational point with probability 0.

3.1.2 $d = 3$

Again for curves of degree 3, any factorization must have a linear term, and so having strength 2 is exactly equivalent to having a rational point. However, this is a much more difficult question for cubics than for conics: a smooth cubic curve is (geometrically) an elliptic curve, and studying rational points on elliptic curves constitutes a huge portion of arithmetic geometry activity.

$\mathbb{F}_p$: Over a finite field, there are now curves of strength 3. For smooth curves E, Hasse's theorem tells us that $\#E(\mathbb{F}_p) = p + a_p + 1$ where $|a_p| \leq 2\sqrt{p}$. Since $2\sqrt{2} < 3$, we in fact have that $\#E(\mathbb{F}_p) > 0$ independent of E and p, so smooth curves are strength 2. For geometrically irreducible singular curves, the singular point still must be rational, but now there is the possibility of a geometrically reducible curve whose components are defined over a proper extension of $\mathbb{F}_p$. For example, the curve $x^3 + 2y^3 + 4z^3 + xyz$ has no $\mathbb{F}_7$-points because geometrically it factors as three lines defined over $\mathbb{F}_{7^3}$, none of which admit $\mathbb{F}_7$-points, and so this gives us an example of a strength 3 cubic.

We can count the number of strength 3 cubics as follows. Given a line $\alpha x + \beta y + \gamma z$ defined over $\mathbb{F}_{p^3}$, this line admits no $\mathbb{F}_p$-points exactly when $\{\alpha, \beta, \gamma\}$ forms a basis for $\mathbb{F}_{p^3}$ as an $\mathbb{F}_p$-vector space. Thus, such lines are in bijection with ordered bases for $\mathbb{F}_{p^3}/\mathbb{F}_p$ up to rescaling by elements of $\mathbb{F}_{p^3}^\times$. To obtain a strength 3 cubic, we multiply the equation for the line by its Galois orbit, so to count strength 3 cubics we must also mod out by the Galois action. There are $\#\,\mathrm{GL}_3(\mathbb{F}_p)$ ordered bases, which we identify in sets of size $3 \cdot \#\mathbb{F}_{p^3}^\times$, so the number of strength 3 cubics is $\frac{(p^3-p)(p^3-p^2)}{3}$. It is also an easy matter to count the number of reducible cubics: counting all pairs (line, conic) overcounts unions of three lines thrice, and unions of a line and a double line twice, so correcting for that we find $\frac{(p^3-1)(p^6-1)}{(p-1)^2} - 2\binom{p^2+p+1}{3} - (p^2+p+1)(p^2+p)$ reducible conics.

$\mathbb{Q}_p$: The density of plane cubics over $\mathbb{Z}_p$ having a $\mathbb{Q}_p$-point was determined by Bhargava, Cremona, and Fisher in [11]. They found that the density is given by $\rho(p) = 1 - f(p)/g(p)$, where f and g are explicit polynomials of degrees 9 and 12 with integer coefficients; in particular, $f(p)/g(p) \sim \frac{1}{3p^3}$ as $p \to \infty$. (Contrast this with the quadratic case, where the convergence of the densities to 1 goes like $1/p$.) It follows from this that the proportion of cubic curves in $\mathbb{Z}[x, y, z]$ with coefficients bounded by B admitting a point over every $\mathbb{Q}_p$ tends to $\prod_p \rho(p)$ as $B \to \infty$. (That the local densities should multiply is not entirely trivial, it is part of Theorem 3.6 in [32].) This product is approximately 97%.

$\mathbb{Q}$**:** For cubic curves, the Hasse principle fails. Selmer gave the example of the cubic curve $3x^3 + 4y^3 + 5z^3$, which has $\mathbb{Q}_p$ points for all p yet has no rational points. In [9], Bhargava showed that a positive proportion of plane cubics over $\mathbb{Q}$ fail the Hasse principle, and a nonzero proportion nontrivially satisfy the Hasse principle (i.e. have points over each $\mathbb{Q}_p$ and over $\mathbb{Q}$). Bhargava conjectures that the proportion of locally soluble cubics which are also globally soluble is exactly 1/3.

3.1.3 $d = 4$

For curves of degree 4, at last we have the possibility of different kinds of strength decompositions. If f is a quartic polynomial, it has three possible kinds of strength 2 decompositions: type (1,1), type (1,2), and type (2,2). The first case corresponds to f having a rational point, the second to an effective rational divisor of degree 2, and the third to an effective rational divisor of degree 4 which is not contained in a line.

$\mathbb{F}_p$**:** We can now have curves with no $\mathbb{F}_p$-points, although they will be uncommon as p gets large. The Lang-Weil estimates [30] imply that a variety over $\mathbb{F}_p$ with an irreducible, geometrically irreducible component will have $\mathbb{F}_p$-points for all sufficiently large p. (For a smooth plane quartic C, the Weil conjectures tell us that $|\#C(\mathbb{F}_p) - (p+1)| \leq 6\sqrt{p}$, so for these curves $p \geq 37$ suffices.) Thus, for sufficiently large p, the only quartics which can fail to have points are geometrically reducible quartics whose components are Galois conjugates defined over a proper extension of $\mathbb{F}_p$. Letting σ be the Frobenius homomorphism, then the possibilities for such a quartic are two quadrics $Q, \sigma Q$ defined over $\mathbb{F}_{p^2}$, or four lines $L, \sigma L, \sigma^2 L, \sigma^3 L$ defined over $\mathbb{F}_{p^4}$. In the first case, Q always has an $\mathbb{F}_{p^2}$-point $\mathbf{a}$, so $[\mathbf{a}] + [\sigma \mathbf{a}]$ gives a degree 2 rational divisor. In the second case, $(L \cap \sigma^2 L) + (\sigma L \cap \sigma^3 L)$ also gives a degree 2 divisor. Thus, such an f always has a strength 2 decomposition of type (1,2).

$\mathbb{Q}_p$**:** Over $\mathbb{Q}_p$, all quartics have strength at most 2, but interestingly the easiest kind of strength decomposition to find is not $(1,1)$. A density 1 set of quartics will be smooth, and for smooth quartic the variety $\mathrm{Sym}^4 C$, which parametrizes degree 4 effective divisors on C, is also smooth. Furthermore, $\mathrm{Sym}^4 C$ has a $\mathbb{Q}_p$-point: the intersection of C with any rational line. Since $(\mathrm{Sym}^4 C)(\mathbb{Q}_p)$ is nonempty, and Sym^C is smooth hence irreducible, the $\mathbb{Q}_p$-points are Zariski dense by the p-adic inverse function theorem. The divisors on C linearly equivalent to the intersection of C with a rational line all map to the class of the canonical divisor under the natural map $\mathrm{Sym}^4 C \to \mathrm{Pic}^4 C$, and hence such divisors are not Zariski-dense in $\mathrm{Sym}^4 C$. Therefore, C must have effective rational divisors of degree 4 not linearly equivalent to the canonical divisor. As noted above, the existence of such a divisor gives a strength 2 decomposition of type (2,2).

$\mathbb{Q}$**:** It is possible to find quartic curves over $\mathbb{Q}$ admitting no effective rational divisors of degree 1 or 2 and such that $(\mathrm{Jac}\, C)(\mathbb{Q}) = \emptyset$. We will show that such curves have strength 3, implying that a local-to-global principle cannot hold in this case. By hypothesis, such a curve does not admit a strength 2 decomposition of type (1,1) or (1,2). As noted in the previous paragraph, having a strength 2 decomposition of type (2,2) is equivalent to having an effective rational divisor D linearly inequivalent to the canonical divisor K. If such a D exists, $D - K$ is a rational point of $\mathrm{Jac}\, C$, so if C satisfies the three conditions above it cannot possess any strength 2 decomposition.

Even among plane quartics with a strength 2 decomposition, types (1,1) and (1,2) should be rare. A smooth degree 4 curve is genus 3, and Faltings' Theorem ([20], Satz 7) tells us that such a curve admits at most finitely many rational points. Indeed, it is conjectured that the asymptotic proportion of curves admitting *any* rational points should be 0 (Conjecture 2.2 of [32]). It should also be rare for a curve to admit an effective rational divisor of degree 2. By another theorem of Faltings, his resolution of the Mordell-Lang Conjecture building on work of Vojta [21, 22, 37], a non-hyperelliptic curve has infinitely many quadratic points only if they are in some sense "parameterized" by positive rank abelian subvarieties of $\mathrm{Jac}\, C$. (See [36] for more on parameterized and isolated points on curves.) It is reasonable to conjecture, along the lines of the conjecture mentioned above, that 0% of quartic curves should have such quadratic points.

Thus, if we accept all these conjectures, we are led to believe that if a quartic curve over $\mathbb{Q}$ has strength 2, the most likely strength 2 decomposition is type (2,2). This is in marked contrast to the geometric situation discussed in *section* 2.2, where the "typical" strength r decomposition is of type (1,...,1). Unfortunately, we do not know how to prove this, so we will record it as a conjecture.

Conjecture 3.1. Ordering plane quartics by height:

- Asymptotically 0% of quartics admit a strength 2 decomposition of type (1,1) or (1,2).
- An asymptotically positive proportion of quartics admit a strength 2 decomposition of type (2,2).

3.2 Strength of Pairs of Conics

We turn now to considering the collective strength of a pair of conics. For a pair of conics over a field k, collective strength of $\{f_1, f_2\}$ is a property of the pencil of conics $sf_1 + tf_2$. Regardless of the ground field k, $\text{str}(f_1, f_2) \leq 2$, as we may always find a linear combination $sf_1 + tf_2$ which has no z^2 term, and such a combination will be contained in the ideal (x, y). Thus, the only thing to determine is when $\text{str}(f_1, f_2) = 1$, which happens exactly when the pencil contains a k-rational fiber reducible over k. The space of all pairs of quadratic polynomials is $\mathcal{P}_2 \times \mathcal{P}_2$, but we wish to consider only those pairs where f_1, f_2 are linearly independent, which is $(\mathcal{P}_2 \times \mathcal{P}_2) \setminus \Delta$, where $\Delta = \{(f, f)\}$ is the diagonal. For brevity, in what follows we will denote $(\mathcal{P}_2 \times \mathcal{P}_2) \setminus \Delta$ by $\mathcal{Q}$.

The reducible fibers of a pencil of conics are controlled by the **base locus**. The base locus of a pencil is the 0-dimensional subscheme of $\mathbb{P}^2$ where f_1 and f_2 both vanish. The base locus of a generic pencil consists of the four distinct geometric points, no three of which are collinear. The reducible fibers of the pencil are exactly of the pairs of lines going through the base locus. In the generic case, there are three such pairs, but as points in the base locus degenerate some of these pairs may become the same, or a union of two lines may become a double line, or both.

Since the base locus is defined over k, the absolute Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ acts on it. Choosing a 0-dimensional length 4 subscheme of $\mathbb{P}^2$ also determines a pencil of conics with that subscheme as its base locus, and this correspondence gives a Galois-equivariant isomorphism between the space of nondegenerate pencils of conics and $\mathrm{Hilb}_4(\mathbb{P}^2)$, the Hilbert scheme of four points in $\mathbb{P}^2$. Thus, we have a map $\mathcal{Q} \to \mathrm{Hilb}_4(\mathbb{P}^2)$ sending a pair to its base locus.

Pencils of conics up to projective equivalence are well-understood. The PGL_3 equivalence classes of pencils stratify the space $\mathrm{Hilb}_4(\mathbb{P}^2)$ of pencils of conics. In Figure 3.1, we enumerate the open strata along with the symbol by which we will refer to each, a description of the base locus of a member of the stratum, and a geometric representative for the family. The subscripts on the $\mathcal{Y}$'s indicate the dimension of the stratum. The families $\mathcal{Y}_4, \mathcal{Y}_4'$, and $\mathcal{Y}_3$ are degenerate families consisting entirely of reducible conics, and for these families we offer a brief description of the family rather than the base locus. Figure 3.2 illustrates the containment poset of these strata. A line is drawn from a larger stratum to a smaller stratum if the former contains the latter in its closure. (Similar diagrams appear in e.g. [25].)

Stratum	Base locus/description	Representative member
$\mathcal{Y}_8$	Four distinct points	$(x(y-z), z(x-y))$
$\mathcal{Y}_7$	Two distinct points and a double point	$(x(y-z), y(x-y))$
$\mathcal{Y}_6$	Two double points	(xz, y^2)
$\mathcal{Y}_6'$	A point and a triple point	$(xz-y^2, xy)$
$\mathcal{Y}_5$	A quadruple point	$(xz-y^2, x^2)$
$\mathcal{Y}_4$	Red. curves through a common point	(x^2, y^2)
$\mathcal{Y}_4'$	Red. curves with a common line	(xy, xz)
$\mathcal{Y}_3$	Lines through a common point union a common line	(x^2, xy)

Figure 3.1: The PGL_3 strata of pencils of conics

Given a pair of conics (f_1, f_2), we can treat $sf_1 + tf_2$ as a quadratic form on the ring $k[s,t]$. If $\mathrm{char}\, k \neq 2$, the geometrically reducible fibers of the pencil correspond to values of s, t where the Gram matrix of this quadratic form drops rank, and these values are

$$\begin{tikzcd}[arrows=dash]
 & \mathcal{Y}_8 \arrow[d] & \\
 & \mathcal{Y}_7 \arrow[dl] \arrow[dr] & \\
\mathcal{Y}_6 \arrow[dr] & & \mathcal{Y}'_6 \arrow[dl] \arrow[dd] \\
 & \mathcal{Y}_5 \arrow[dl] & \\
\mathcal{Y}_4 \arrow[dr] & & \mathcal{Y}'_4 \arrow[dl] \\
 & \mathcal{Y}_3 &
\end{tikzcd}$$

Figure 3.2: The containment structure of the strata closures

detected by the determinant of the Gram matrix. We will call this the determinant of (f_1, f_2), and denote it by $\det(f_1, f_2)$; it is a homogeneous cubic polynomial in s, t. We may write $\det(f_1, f_2)$ explicitly: if $sf_1 + tf_2 = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$, where the $a_i = a_i(s,t)$ are linear expressions in s and t, then the determinant of the Gram matrix is $4a_1a_2a_3 - a_4a_5a_6 + a_1a_6^2 + a_2a_5^2 + a_3a_4^2$.

The following lemma gives a useful criterion for identifying strength 1 pencils based on the Galois action. This criterion will be used repeatedly in the remainder of the section.

Lemma 3.2. Let $f_1, f_2 \in k[x, y, z]$ be conics, and let $B \subset \mathbb{P}^2$ be their base locus. Then $\operatorname{str}(f_1, f_2) = 1$ if and only if there is a length 2 subscheme $X \subset B$ which is preserved by the action of G_k.

Proof. Suppose $\operatorname{str}(f_1, f_2) = 1$, so there is $s, t \in k$ so that $sf_1 + tf_2 = gh$ for two linear forms $g, h \in k[x, y, z]$. Then $X = V(g) \times_{\mathbb{P}^2} B$ is a length 2 subscheme of B, and since g has k-rational coefficients, X must be preserved by G_k.

On the other hand, suppose G_k preserves such an X. Then there is a unique line L_1 in $\mathbb{P}^2$ passing through X, and that line is preserved by G_k, hence is defined over k. There is also a unique line L_2 so that the union $L_1 \cup L_2$ contains B as a subscheme, and L_2

is likewise defined over k. Thus, $L_1 \cup L_2$ is a reducible fiber of the pencil which is also defined over k. $\square$

It is immediate from Lemma 3.2 that a pencil of conics over $\mathbb{R}$ always has strength 1. The absolute Galois group $G_{\mathbb{R}}$ is cyclic of order 2, so indeed the Galois orbit of a closed point in the base locus must have degree 1 or 2.

We collect our various results about pencils of conics in the following multi-part theorem.

Theorem 3.3. Let k be a number field, a finite field, or a nonarchimedean local field.

(a) If k is a number field, then consider the set of all pairs (f_1, f_2) with coefficients of height at most B. Let $N(B)$ be the total number of such pairs, and let $N_{\mathrm{str}=1}(B)$ be the number with strength 1. Then $N_{\mathrm{str}=1}(B)/N(B) = O((B^{[k:\mathbb{Q}]/2} \log B)^{-1})$.

(b) If $k = \mathbb{F}_q$ if a finite field, then the number of pencils of conics in the strength 2 locus of $\mathrm{Hilb}_4(\mathbb{P}^2)$ is $\frac{7}{12} \cdot \# \mathrm{PGL}_3(\mathbb{F}_q)$.

(c) If k is a nonarchimedean local field, let $\mathcal{O}_k$ be the ring of integers, $\mathfrak{p}$ the maximal ideal, $\mathcal{O}_k/\mathfrak{p} = \mathbb{F}_q$ the residue field, and μ the Haar measure normalized so that $\mu(\mathcal{O}_k) = 1$. The measure of the set of pencils (f_1, f_2) with $f_i \in \mathcal{O}_k$ and $\mathrm{str}(f_1, f_2) = 2$ differs from $\frac{7}{12} \cdot \frac{\# \mathrm{PGL}_3(\mathbb{F}_q)}{\# \mathrm{Gr}(2,6)(\mathbb{F}_q)}$ by $O(q^{-2})$.

Proof of Theorem 3.3(a). We will show the stronger statement that the number of pairs such that the pencil has a rational fiber which is geometrically reducible satisfies the above asymptotic. The determinant form $\det(f_1, f_2)$ can be viewed as a polynomial in $k[s, t, a_1, \ldots, a_6, b_1, \ldots, b_6]$, where the a_i are the coefficients of f_1 and the b_i are the coefficients of f_2. We would like to show that for a general choice of the a_i and b_i, the resulting polynomial in $k[s, t]$ is irreducible with Galois group S_3, and then quantitative forms of Hilbert's irreducibility theorem will give us the result. To determine the Galois group, it is enough to choose a judicious example pair (f_1, f_2), since the generic Galois group contains the Galois group of any particular fiber as a subgroup. Let $\mathcal{O}_k$ be the ring of integers, and let $\pi \in \mathcal{O}_k$ generate a maximal ideal, and let $f_1 = x^2 - \lambda\pi y^2 + yz$ and $f_2 = -\pi y^2 + xz$ for some λ yet to be determined. Then $\det(f_1, f_2) = s^3 - \pi\lambda s t^2 - \pi t^3$. This polynomial is π-Eisenstein, hence irreducible. It will have Galois group S_3 as long as its discriminant $4\pi^3\lambda^3 - 27\pi^2$ is not a square in k. Thus, we succeed as long as λ is chosen so that there is no k-rational solution to $\mu^2 = 4\pi^3\lambda^3 - 27\pi^2$. This is an elliptic curve in λ and μ, so by Siegel's theorem it has only finitely many integer points. If λ is chosen in $\mathcal{O}_k$ outside that finite set, then there will be no such μ, and hence we will obtain an example where the Galois group of $\det(f_1, f_2)$ is S_3.

Now, by a quantitative form of Hilbert's irreducibility theorem [38], we obtain that the number of pairs of height at most B for which the Galois group is smaller that S_3 is $O(B^{[k:\mathbb{Q}](n+1/2)} \log B)$, while the total number of pairs is $\asymp B^{[k:\mathbb{Q}](n+1)}$. This gives the desired result. □

Remark 3.4. Our argument above leaves open the possibility that the true asymptotic number of strength 2 pairs has an even smaller exponent. We leave it for further work to find the optimal exponent, and to investigate whether anything can be said about the implied constant.

Before we discuss the case over $\mathbb{F}_q$, we briefly mention how the strata $\mathcal{Y}_i$ decompose under the action of $G_{\mathbb{F}_q}$. This is determined completely by the Galois action on the base locus, and there are relatively few ways for a procyclic group to act on a set with at most four elements. We follow [10], which also discusses pencils of conics over $\mathbb{F}_q$, although for different ultimate ends. We can characterize each Galois orbit by its "splitting type", that is to say, the degrees f_i of the field extension over $\mathbb{F}_q$ where the orbit splits, and the multiplicity e_i of the orbit. We represent this data by the symbol $(f_1^{e_1} f_2^{e_2} \ldots)$. Thus, for example, (1111) represents a base locus consisting of four distinct $\mathbb{F}_q$-rational points (which belongs to $\mathcal{Y}_8$), while $(1^2 2)$ represents a base locus consisting of a rational point with multiplicity 2 and a conjugate pair of quadratic points (which belongs to $\mathcal{Y}_7$). There are eleven symbols possible among non-degenerate strata of pencils: (1111), (112), (13), (22), (4), $(1^2 11)$, $(1^2 2)$, $(1^2 1^2)$, (2^2), $(1^3 1)$, and (1^4). We quote [10] for the number of polynomials of each splitting type.

Lemma 3.5 ([10] Lemma 21)**.** The number of pencils of splitting type:

- (1111) is $\frac{1}{24}\#\,\mathrm{PGL}_3(\mathbb{F}_q) = (q-1)^2q^3(q+1)(q^2+q+1)$;
- (112) is $\frac{1}{4}(q-1)^2q^3(q+1)(q^2+q+1)$;
- (13) is $\frac{1}{3}(q-1)^2q^3(q+1)(q^2+q+1)$;
- (22) is $\frac{1}{8}(q-1)^2q^3(q+1)(q^2+q+1)$;
- (4) is $\frac{1}{4}(q-1)^2q^3(q+1)(q^2+q+1)$;
- (1^211) is $\frac{1}{2}(q-1)q^3(q+1)(q^2+q+1)$;
- (1^22) is $\frac{1}{2}(q-1)q^3(q+1)(q^2+q+1)$;
- (1^21^2) is $\frac{1}{2}q^3(q+1)^2(q^2+q+1)$;
- (2^2) is $\frac{1}{2}(q-1)q^3(q^2+q+1)$;
- (1^31) is $(q-1)q^2(q+1)(q^2+q+1)$;
- (1^4) is $(q-1)q(q+1)(q^2+q+1)$.

We now proceed to the proof of Theorem 3.3(b).

Proof of Theorem 3.3(b). By Lemma 3.2, we can tell if a pencil of conics is strength 1 by whether the base locus contains a length 2 subscheme preserved by the Galois action. We can determine that from the splitting type of the pencil: if the pencil has either two rational points in the base locus (i.e. two 1s in the splitting type symbol), a rational point of multiplicity 2 or greater (a 1^2), or a quadratic point (a 2), then that constitutes a subscheme of length 2 fixed by Galois. Looking at the above enumeration of splitting types for nondegenerate pencils, we see the only pencils which fail to have either of these are splitting types (13) and splitting type (4). Together, these have $\frac{7}{12} \cdot \#\,\mathrm{PGL}_3(\mathbb{F}_q)$ polynomials between them. □

Proof of Theorem 3.3(c). Let k be as in the statement of Theorem 3.3(c). If the pair (f_1, f_2) has strength 1 over k, it certainly must have strength 1 over the residue field $\mathbb{F}_q$. Thus, we can always lift a high strength (strength 2) decomposition from $\mathbb{F}_q$ to k, but we may not be able to lift a strength 1 decomposition (a factorization).

Suppose we have a pair (f_1, f_2) over $\mathcal{O}_k/\mathfrak{p}$ with collective strength 1, so there exists s, t, g, h satisfying $sf_1 + tf_2 = gh$. By a deformation of (f_1, f_2) mod $\mathfrak{p}^m$ we mean a pair (F_1, F_2) over $\mathcal{O}_k/\mathfrak{p}^m$ such that $F_i \equiv f_i \pmod{\mathfrak{p}}$ for $i = 1, 2$. Given a deformation of (f_1, f_2) mod $\mathfrak{p}^m$, we wish to know when we can find deformations S, T, G, H so that $SF_1+TF_2 = GH \pmod{\mathfrak{p}^m}$. If we can do this for all m, then the strength 1 decomposition of f_1, f_2 lifts to a strength 1 decomposition of any pair over $\mathcal{O}_k$ reducing to f_1, f_2 modulo $\mathfrak{p}$.

Up to replacing (f_1, f_2) by a different pair representing the same pencil, we may assume that f_1 is one of the singular fibers of the pencil, i.e. we may assume that $s = 1$ and $t = 0$. Further, up to the action of PGL_3 on the base locus, we may assume that the singular fiber f_1 factors either as xy or as x^2, i.e. that $g = x$ and $h = x$ or y.

We begin by considering the case where $f_1 = xy \pmod{\mathfrak{p}}$. Given a deformation (F_1, F_2) mod $\mathfrak{p}^2$, let us see what is required to lift this factorization. Let π be a uniformizer for $\mathfrak{p}$, and write $S = 1 + \pi s_1$, $T = \pi t_1$, $G = x + \pi\ell_1$, $H = y + \pi\ell_2$, where $s_1, t_1 \in \mathbb{F}_q$ and ℓ_1, ℓ_2 are linear forms yet to be determined. Expanding, we get

$$\begin{aligned}
SF_1 + TF_2 &= GH \pmod{\mathfrak{p}^2} \\
(1 + \pi s_1)F_1 + \pi t_1 F_2 &= (x + \pi\ell_1)(y + \pi\ell_2) \quad (\mathfrak{p}^2) \\
(F_1 - xy) + \pi(s_1F_1 + t_1F_2) &= \pi(y\ell_1 + x\ell_2) \quad (\mathfrak{p}^2) \\
F_1 - xy &= \pi(y\ell_1 + x\ell_2 - s_1F_1 - t_1F_2) \quad (\mathfrak{p}^2).
\end{aligned}$$

Dividing through by π , we obtain

$$\frac{F_1 - xy}{\pi} = y\ell_1 + x\ell_2 - s_1xy - t_1f_2 \pmod{\mathfrak{p}}.$$

Note that $F_1 = xy \pmod{\mathfrak{p}}$, so the left hand side makes sense. Define a linear operator $L\colon S_1(\mathbb{F}_q)^2 \times \mathbb{F}_q^2 \to S_2(\mathbb{F}_q)$ by $L(\ell_1, \ell_2, s_1, t_1) = y\ell_1 + x\ell_2 - s_1xy - t_1f_2$. Then we succeed in lifting mod $\mathfrak{p}^2$ exactly when $(F_1 - xy)/\pi$ is in the image of L. Note that if f_2 has a nonzero z^2 coefficient, then L is onto. Thus, the lifting can only fail if $f_2 \in (x, y)$, which only happens if the base locus of the pencil contains $(0, 0, 1)$ with multiplicity greater than 1. In that case, it is probability $1/q$ that we succeed in lifting to $\mathfrak{p}^2$.

Suppose we have succeeded in lifting mod $\mathfrak{p}^{m-1}$, so we have a deformation (F_1', F_2') and we have an equality $S'F_1' + T'F_2' = G'H' \pmod{\mathfrak{p}^{m-1}}$. Given a deformation (F_1, F_2) of (F_1', F_2') mod $\mathfrak{p}^m$, a computation essentially identical to the one above shows that we can continue lifting exactly when $(F_1 - G'H')/\pi^{m-1}$ is in the image of L. Therefore, by the preceding discussion, if $f_2 \notin (x, y)$, we will always succeed in lifting.

Turning now to the case where $f_1 = x^2 \pmod{\mathfrak{p}}$, we can perform a computation like above to conclude that we succeed in lifting mod $\mathfrak{p}^2$ if and only if $(F_1 - x^2)/\pi = x(\ell_1 + \ell_2) - s_1x^2 + t_1f_2$ has a solution mod $\mathfrak{p}$. Again, this is an assertion that $(F_1 - x^2)/\pi$ is in the image of some linear operator, but that operator is now never onto. Thus, in this case it is only probability $1/q$ that we succeed in lifting mod $\mathfrak{p}^2$.

Taken together, what we have shown is that:

i. if a singular fiber of a given pencil is a union of distinct line, and the singular point of the union is not in the base locus of the pencil, then that union certainly lifts to k;

ii. if a singular fiber is a union of distinct lines and the singular point is in the base locus, or if a singular fiber is a line with multiplicity 2, then lifting may or may not succeed, and just lifting mod $\mathfrak{p}^2$ is a probability $1/q$ event.

This is not enough to determine the exact density of pairs (f_1, f_2) over $\mathcal{O}_k$ of strength 1, but it is enough to get an approximation. Notice that any strength 1 pencil contained in $\mathcal{Y}_8$ or $\mathcal{Y}_7$ has a singular fiber satisfying condition i. above (as does a pencil in $\mathcal{Y}_6$ of splitting type (1^21^2)). Thus, the density of strength 1 pencils which lift to strength 2 pencils over $\mathcal{O}_k$ is at most

$$\frac{1}{2} \cdot \frac{(q-1)q^2(3q+2)(q^2+q+1)}{\#\mathbb{P}^8(\mathbb{F}_q)} = O(q^{-2}).$$

$\square$

Remark 3.6. It is dissatisfying not to be able to exactly compute the density of pencils with strength 2. We attempted to study the problem of higher lifting in the problematic case ii. above, hoping to find a recursive formula which would compute the exact density. Unfortunately, the complications became ever more fine, and the number of cases under consideration rapidly overwhelmed us. It may be possible, but it seems quite daunting if it is.

Remark 3.7. The results of Theorem 3.3 leave open the possibility of a local-to-global principle for pencils of conics of the following form: "a pencil of conics over a number field k has strength 1 if and only if it has strength 1 over every completion k_v of k." Although we were unable to compute the exact densities, our approximation suffices to conclude that as the residue characteristic grows the probability of being strength 1 over k_v approaches $\frac{5}{12}$. Thus, the probability of being everywhere locally strength 1 will tend to 0, consistent with our result in Theorem 3.3(a).

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