Always Use Pascal!

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1 Introduction

The binomial theorem states that for any a, b in a commutative ring,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Most students who have completed a course in discrete mathematics will have either proven or seen a proof of this equality. Although the coefficients on the right side of the above equality are known as *binomial coefficients*, owing to their association with this theorem, students usually encounter them first as a shorthand notation for a counting problem: if S is a set of n elements, how many distinct subsets of S have k elements? Most students will also be familiar with the factorial expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

I really clung to the factorial expression for binomial coefficients when I first learned it, as I think many do. After all, the factorial expression gives you a way to actually *calculate* the binomial coefficients – albeit a tedious one, but a concrete one. If you're a student concerned primarily with perhaps having to calculate the answer if someone should ask you "how many different groups of four can I make in my class of 25 5th graders?", then the ability to have a concrete tool of calculation is I think very appealing. However, in the manipulation of binomial coefficients themselves, the factorial expression is unwieldy. Much more elegant is the Pascal identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The Pascal identity is usually proven with a counting argument, though it can also be derived from the factorial representation. (Somewhat more interesting is deriving the factorial representation from the Pascal identity.) The fact that the Pascal identity is a recursive formula is very powerful, as it makes it possible to prove things about binomial coefficients inductively. Trying to prove these things using the factorial representation would be ridiculous. My goal here is to try to demonstrate a few such proofs, and to convince the reader of the utility of the Pascal identity.

2 Alternating sums of binomial coefficients

The binomial theorem allows us to deduce certain interesting things by taking particular values of a, b. For example, taking $a = b = 1$ we get

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

That is, the total number of subsets of an n element set is 2^n. This is a fact that we could have derived through a counting argument, but it's at least reassuring to know that our theorem also gives us the correct answer in this case. Another question of interest is how many subsets of an n element set, S, have an odd or even number of elements. Certainly if n is odd, and if $K \subset S$ is of size k, then $S \setminus K$ has size $n - k$, which is odd when k is even and visa versa. So we see that at least when n is odd there is a bijection between the sets of odd size and the sets of even size.

However, it doesn't seem immediately obvious that a similar thing should hold for even n. Thankfully we have the binomial theorem at our disposal, as if we take $a = 1, b = -1$ we obtain

$$(1-1)^n = 0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \tag{1}$$

We can further divide this identity by splitting up the right side to obtain

$$\sum_{\substack{k \le n \\ k \text{ even}}} \binom{n}{k} - \sum_{\substack{k \le n \\ k \text{ odd}}} \binom{n}{k} = 0.$$

The first sum represents the number of subsets of even size, and the second sum the number of subsets of odd size, so this is exactly the statement that there are as many of one as there are of the other. Again we get an answer by considering a sum of binomial coefficients.

The sum in (1) is quite a bizarre expression to encounter without the context of the binomial theorem. Let's try to understand it without appealing to the binomial theorem for a moment. This sum is traversing a row in Pascal's triangle, alternating signs each time. How exactly does this sum end up being zero? By examining Pascal's triangle, notice that until

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Figure 1: Pascal's Triangle

we reach the middle of each row, the numbers increase from left to right. So, that means the absolute value of any partial sum is at most the absolute value of the most recently added entry, up until we get to the middle of the row. For example, in row 6 the first few partial sums are

(0) 1,

(1) $1 - 6 = -5$,

(2) $1 - 6 + 15 = 10$,

(3) $1 - 6 + 15 - 20 = -10$.

Moreover, because the absolute values remain relatively small, and the absolute value of the summands increases, the sequence of partial sums alternates up to the middle of the row. It would make sense for the partial sums to continue to alternate from nonnegative to nonpositive, and after the middle of each row decrease in value, until they slowly hone in on 0.

How should we look to make this intuition precise? Well, when we're considering partial sums of (1), we are considering sums of the form

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k},$$

for some m, $0 \leq m \leq n$. Let's gather a little more data first. We know from (1) that when $m = n$, the sum is 0, so let's only consider the $m < n$. Let $s(m)$ represent the mth partial sum, explicitly given above. Below I summarize some quick calculations of these $s(m)$'s.

n	m	$s(m)$
1	0	1
2	0	1
2	1	-1
3	0	1
3	1	-2
3	2	1
4	0	1
4	1	-3
4	2	3
4	3	-1

These values perhaps looks somewhat familiar. A more illuminating format for them might be to put them in a table, where the rows correspond to our n, and the columns to our m. If we do, we'll end up with something that looks like:

1

1 -1

1 -2 1

1 -3 3 -1

At this point it is clear as daylight what we are looking at: this is the Pascal triangle but alternating sign! Although, we must be a little careful here – the first row of this triangle is the one associated with $n = 1$, whereas in the usual triangle it is associated with $n = 0$. Nevertheless, we can now exactly formulate a proposition about the partial sums of 1.

Proposition. For $n > 0$, and any $m \geq 0$,

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \tag{\dagger}$$

Here, one is certainly invited to attempt this calculation by means of the factorial expression, and perhaps for calculating specific values that would be sufficient. However, the general case is almost certainly intractible in that form, and at the very least extraordinarily ugly. Let us see what work we can do with the Pascal recurrence.

Proof of Proposition 2. We proceed by induction on n. First of all, we can restrict our attention to $m < n$, as when $m \geq n$ the right side is zero by definition, and expression (1) tells us the left side will also be zero. Now, for our base case take $n = 1$ and we only have to consider $m = 0$. In the preceding table, the value we calculated was $1 = (-1)^0 \binom{0}{0}$, so our theorem holds for all $m \geq 0$ when $n = 1$. Suppose we have ($\dagger$) for all m and some n. By the Pascal identity, we have

$$\begin{aligned}
\sum_{k=0}^{m} (-1)^k \binom{n+1}{k} &= \sum_{k=0}^{m} (-1)^k \left(\binom{n}{k} + \binom{n}{k-1} \right) \\
&= \sum_{k=0}^{m} (-1)^k \binom{n}{k} + \sum_{k=0}^{m} (-1)^k \binom{n}{k-1}.
\end{aligned}$$

In the second sum above, notice that the term corresponding to $k = 0$ vanishes. Factoring out a -1 and reindexing, we rewrite the above as

$$\begin{aligned}
\sum_{k=0}^{m} (-1)^k \binom{n}{k} - \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} &= (-1)^m \binom{n-1}{m} - (-1)^{m-1} \binom{n-1}{m-1} \qquad \text{(by hypothesis)} \\
&= (-1)^m \left(\binom{n-1}{m} + \binom{n-1}{m-1} \right).
\end{aligned}$$

One final application of Pascal's identity shows that this final expression is equal to $(-1)^m \binom{n}{m}$, as desired. $\square$