Module Theory 3

Localizing modules, tensor products, projective and flat modules, a bit about algebras

by Ivan Aidun

These are notes interspersed with exercises. The purpose of these notes is to be more fleshed out than Evan Dummit's old notes, but shorter and more focused than a textbook treatment. Halfway between Dummit and Dummit and Foote, so to speak. I hope these are helpful to you!

A long-awaited theorem on $M_2(\mathbb{C})$-modules

Way back in Module Theory 1, I claimed that every f.g. left $M_2(\mathbb{C})$-module was of the form $(\mathbb{C}^2)^n$ for some n, where the action of a matrix M was "coordinate-wise" on the n different 2×1 vectors. I meant to include a section about that in Module Theory 1, but then those notes got too long, then I forgot while working on Module Theory 2, so it is going here now.

Exercise 1. Write $V = \mathbb{C}^2$, considered as a left $M_2(\mathbb{C})$-module via matrix-vector multiplication. Recall that previously you proved that V was a simple module, and that V is the only simple left $M_2(\mathbb{C})$-module up to isomorphism.

(a) Let M be any left $M_2(\mathbb{C})$-module. Prove that the image of any homomorphism $V \to M$ must either be 0 or a submodule of M isomorphic to V.

(b) Prove that if $N_1, N_2 \subset M$ are any two distinct submodules of M isomorphic to V, then either $N_1 = N_2$ or $N_1 \cap N_2 = \{0\}$.

(c) Prove that $M_2(\mathbb{C}) \cong V \oplus V$ as a left module over itself.

(d) Conclude that any free left module $M_2(\mathbb{C})^{\oplus k}$ is isomorphic to $V^{\oplus 2k}$. (Here, $A^{\oplus n}$ means the direct sum of n copies of A. I apologize that the notation is weird and somewhat awkward, it is standard in order to distinguish between taking direct sums and tensor powers.)

(e) Let M be any finitely-generated $M_2(\mathbb{C})$-module. Prove that M is the internal direct sum of finitely many submodules isomorphic to V.

Localizing modules

Just as we can localize rings, we can also localize modules. Starting with a ring R, an R-module M, and a multiplicatively closed subset $S \subset R$, we can form $S^{-1}M$, which will be an $S^{-1}R$-module. The process of doing so is pretty much the same as for forming the ring $S^{-1}R$ in the first place:

- The elements of $S^{-1}M$ are symbols of the form $\frac{m}{s}$, where $m \in M$ and $s \in S$.
- Two symbols $\frac{m_1}{s_1}, \frac{m_2}{s_2}$ are equal if their cross difference is S-torsion, that is, if there exists $s \in S$ so that $s(s_2 m_1 - s_1 m_2) = 0$.

- The addition in $S^{-1}M$ is defined by taking common denominators

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}.$$

- The $S^{-1}R$-module multiplication is probably what you expect: $\frac{r}{s_1} \cdot \frac{m}{s_2} = \frac{rm}{s_1 s_2}$.

Just like with rings, if S is the compliment of a prime ideal $\mathfrak{p}$, then the $R_{\mathfrak{p}}$-module $S^{-1}M$ is usually written as $M_{\mathfrak{p}}$. I am not sure that I've ever seen special notation for a module over a localization of the form $R_f = R[\frac{1}{f}]$.

Exercise 2. Consider the $\mathbb{Z}$-module $\mathbb{Z}/12\mathbb{Z}$.

(a) Let S be the set of powers of 2, so $S^{-1}\mathbb{Z} = \mathbb{Z}[\frac{1}{2}]$. What is the isomorphism class of the $\mathbb{Z}[\frac{1}{2}]$-module $S^{-1}\mathbb{Z}/12\mathbb{Z}$? (Hint: look back at Ring Theory 2, Exercise 10.)

(b) Now let S be the set of powers of 3. What is the isomorphism class of the $\mathbb{Z}[\frac{1}{3}]$-module $S^{-1}\mathbb{Z}/12\mathbb{Z}$?

(c) Show that if S contains no powers of 2 or 3, then $S^{-1}\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$.

The module $S^{-1}M$ is *also* an R-module, because we have the localization homomorphism $R \to S^{-1}R$ (see Module Theory 1, Example 4). However, being an $S^{-1}R$ module is a more special property.

Exercise 3. Prove that an $S^{-1}R$-module is the same thing as an R-module M such that for every $s \in S$, the "multiplication by s" map $s\cdot\colon M \to M$ is an isomorphism.

The localization homomorphism and universal property

As with localizing rings, we get a homomorphism $\ell\colon M \to S^{-1}M$, also called the "localization homomorphism", by sending $m \mapsto \frac{m}{1}$. The homomorphism ℓ is a homomorphism *of R-modules*, so $\ell(rm) = r\ell(m)$ for all $r \in R$.

Exercise 4. This exercise generalizes the statement that the localization homomorphism $R \to S^{-1}R$ is injective iff S contains no zero-divisors.

(a) Let M be an R-module. Take some $m \in M$, and let $\mathfrak{a} = \operatorname{Ann}(m)$. Prove that $\ell(m) = 0$ in $S^{-1}M$ if and only if $S \cap \mathfrak{a} \neq \emptyset$.

(b) Prove that ℓ is injective if and only if M has no s-torsion for any $s \in S$.

(c) Prove that $S^{-1}M$ has no s-torsion for any $s \in S$.

(d) Explain your answers to Exercise 2.

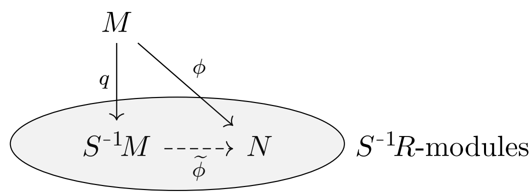
It is common to hear people paraphrase this by saying that localization *kills torsion*.

The localization homomorphism has a pretty similar universal property to the one for rings, but it's worth stating on its own.

Theorem. Let R be a ring, M an R-module, S a multiplicatively closed subset of R, and $\ell\colon M \to S^{-1}M$ be the localization homomorphism. Let N be an $S^{-1}R$-module, and recall that N is also an R-module. If $\phi\colon M \to N$ is an R-module homomorphism, then there exists a unique function $\widetilde{\phi}\colon S^{-1}M \to N$ such that $\phi = \widetilde{\phi} \circ \ell$. The map $\widetilde{\phi}$ will be a homomorphism of $S^{-1}R$-modules.

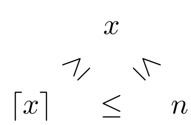
Concretely, the map $\widetilde{\phi}$ sends the fraction $\frac{m}{s} \mapsto \frac{\phi(m)}{s}$.

I mention this explicitly because, in addition to carrying all the interpretation of previous universal properties that we have seen, this universal property sort of connects two different worlds: the world of R-modules and the world of $S^{-1}R$-modules. This is illustrated in the diagrammatic interpretation below, where I have circled and labelled the $S^{-1}R$-modules.



A restatement of the theorem is "for any R-module homomorphism ϕ from M to an $S^{-1}R$-module N, there exists a unique $S^{-1}R$-module homomorphism $\widetilde{\phi}$ making the above diagram commute". You will sometimes hear people say that a universal property like this is a kind of "approximation theorem". We already said that being an $S^{-1}R$-module is more restrictive than being an R-module, but we can think of $S^{-1}M$ as being the $S^{-1}R$-module that "best approximates" M.

Here's a comparison: consider the ceiling function $\lceil * \rceil\colon \mathbb{R} \to \mathbb{Z}$. It has the property that whenever n is an integer and $n \geq x$, then $n \geq \lceil x \rceil$. This shows that among {integers greater than x}, $\lceil x \rceil$ is the best approximation to the real number x. I could similarly make a diagram:



Localization and exact sequences

I'm going to tell you the payoff of this section up front, to help motivate why the rest of the section exists. Suppose I want to understand the $\mathbb{C}[x, y]$-module $(\mathbb{C}[x, y]/(xy))_x$. To spell it out, first I take the ring $\mathbb{C}[x, y]$ and mod out by the ideal (xy), and then I take that ring and I localize by inverting the element x. What I'd really like to say is that I can switch the order of the localization and the quotient, because if I could I could compute

$$\begin{aligned}\left(\frac{\mathbb{C}[x,y]}{(xy)}\right)_x &\cong \frac{\mathbb{C}[x,y]_x}{(xy)_x} \\ &\cong \frac{\mathbb{C}[x,y]_x}{(y)_x} \\ &\cong \left(\frac{\mathbb{C}[x,y]}{(y)}\right)_x \\ &\cong \mathbb{C}[x]_x \\ &= \mathbb{C}[x, x^{-1}].\end{aligned}$$

Here, the second line follows from the first because in $\mathbb{C}[x,y]_x$, x is a unit, so the ideal generated by y is the same as the ideal generated by xy. We have swapped the order of localization and quotients twice, once in the first line, and once going from the second line to the third.

Hopefully you see why being able to do these manipulations is appealing. In this albeit contrived circumstance, it allows us to think about the ring $\mathbb{C}[x, x^{-1}]$, which seems much easier to understand than $(\mathbb{C}[x,y]/(xy))_x$. And I promise that in number theory, commutative algebra, and algebraic geometry, computations like this come up not that infrequently.

What exactly would it take to prove something like this? If you read the title of this section, you might have an inkling about what my answer is, but I do genuinely want you to pause to think about how you might justify the manipulations I was doing above. I think that I maybe could do it straight from the definitions, but it would be an absolute slog. Luckily, there is a better way, thought of by very clever people. Here is the first place that being conversant in exact sequences, and being willing to draw lots of diagrams, will really streamline our ability to do algebra.

Whenever we have a homomorphism of R-modules $f\colon A \to B$, we can compose f with the localization map $\ell_B\colon B \to S^{-1}B$. Concretely, the composition map $\ell_B \circ f\colon A \to S^{-1}B$ sends $a \mapsto \frac{f(a)}{1}$. Now, the composition will be a map from A to an $S^{-1}R$-module, so by the universal property there will exist an $S^{-1}R$-module homomorphism so that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\ell_A} & & \downarrow{\ell_B} \\ S^{-1}A & \overset{\widetilde{f}}{\dashrightarrow} & S^{-1}B \end{array}$$

I said it way back in Ring Theory 2, so I'll remind you that a diagram **commutes** if composing the homomorphisms along each directed path gives the same thing in the end. In this diagram, that means that $\widetilde{f} \circ \ell_A = \ell_B \circ f$.

Now, if instead of just one homomorphism we had a sequence of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we would get an induced sequence of homomorphisms

$$S^{-1}A \xrightarrow{\tilde{f}} S^{-1}B \xrightarrow{\tilde{g}} S^{-1}C.$$

The important question here turns out to be "if the original sequence of homomorphisms was exact, is the induced sequence exact as well?".

Exercise 5. (localization preserves exact sequences) Prove that if the original sequence of homomorphisms was exact, the induced sequence is exact as well. (Hint: I am asking you to prove two sets are equal, so first prove one containment, then the reverse containment.)

This immediately gives us the justification we were looking for in the first place. If

$$0 \to A \to B \to C \to 0$$

is a SES of R-modules, then by Exercise 5

$$0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0$$

is a SES of $S^{-1}R$-modules. The first SES tells us that $C = B/A$, so $S^{-1}C = S^{-1}(B/A)$ is the order "quotient first, then localize". On the other hand, the second SES tells us that $S^{-1}C = S^{-1}B/S^{-1}A$, which is the order "localize first, then quotient".

Comment. I am not going to use any words coming from the theory developed by Eilenberg and Mac Lane in these notes, but I will note that people usually will say "localization is exact" rather than saying "localization preserves exact sequences".

Checking properties locally

One final important feature of localizing modules is that it gives us a way to check properties "locally", the way we might in topology. Here's a couple examples to get us started.

Exercise 6. (being the zero module is a local property) Prove that the following are equivalent:

(i) $M = 0$.

(ii) For every multiplicative subset $S \subset R$, $S^{-1}M = 0$.

(iii) For every prime ideal $\mathfrak{p} \subset R$, $M_{\mathfrak{p}} = 0$.

(iv) For every maximal ideal $\mathfrak{m} \subset R$, $M_{\mathfrak{m}} = 0$.

Exercise 7. Suppose $f\colon M \to N$ is an R-module homomorphism.

(a) Prove that f is surjective if and only if for all maximal ideals $\mathfrak{m} \subset R$, $\widetilde{f}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is surjective.

(b) Prove the same statement with the word "surjective" replaced by "injective".

Exercise 8. Choose either "injective" or "surjective". Give an example of a homomorphism between finitely generated $\mathbb{Z}$-modules $f\colon M \to N$ so that

(i) f is not whichever adjective you chose.

(ii) For all but one maximal ideal $\mathfrak{m} \subset \mathbb{Z}$, $\widetilde{f}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ *is* that adjective.

I know the first time I learned these facts, they bothered me because it seemed like they "reduced" an easy problem like checking if a homomorphism is injective/surjective to the hard problem of checking infinitely many homomorphisms of much more opaquely-defined modules to see if they are injective/surjective. The utility of statements like these is that what will usually happen is that most of the maps $\widetilde{f}$ will automatically have whatever property, so you will only need to check like one or two actual maximal ideals $\mathfrak{m}$. Furthermore, altho I don't have enough space to convince you here, modules over local rings have lots of nice properties that modules over general rings do not, so it actually can be an improvement to trade one global computation for several local computations.

Tensor products and flat modules

We come now to the big doozy. I will give it my best shot, but I'm not sure I can do better than Keith Conrad.

Tensor products over commutative rings

I'm going to start with the commutative case, because it is easier to explain and more common. Given two R-modules M, N, we can form their **tensor product**, notated $M \otimes_R N$, which will again be an R-module.

- As a set, $M \otimes_R N$ consists of symbols of the form $m \otimes n$, where $m \in M$ and $n \in N$, as well as all finite R-linear combinations of these elements. These symbols are subject to the following two relationships.

- For all $m_1, m_2 \in M$, $n \in N$, we have $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, and similarly on the right.

- For all $r \in R$, $(rm) \otimes n = m \otimes (rn) = r(m \otimes n)$.

The symbols $m \otimes n$ are called "simple tensors", and by definition they span the tensor product $M \otimes_R N$. As briefly mentioned in the Linear Algebra notes, the simple tensors can be thought of as representing some sort of "general product" between the elements $m \in M$ and $n \in N$, constrained only to behave like an R-bilinear map would.

Facts.

- Every simple tensor of the form $0 \otimes n$ or $m \otimes 0$ is equal to the zero element of the the tensor product $M \otimes_R N$.
- There is an isomorphism $M \otimes_R N \cong N \otimes_R M$ that just swaps all the simple tensors, so it doesn't really matter what order you take the tensor product in. (But probably don't swap on the fly in the middle of a paper or something.)

Example 1. Take the k-vector spaces $V = k^m$, $W = k^n$. Give V the basis e_i, and W the basis f_j. Then $V \otimes_k W \cong k^{mn}$, with a basis given by the simple tensors $e_i \otimes f_j$.

Example 2. Consider the tensor product $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$. We have already said that $0 \otimes n = 0 \otimes 0$ for all $n \in \mathbb{Z}/3\mathbb{Z}$. On the other hand, since $1 \equiv 3 \pmod 2$, we have $1 \otimes n = 3 \otimes n = 1 \otimes 3n = 1 \otimes 0 = 0 \otimes 0$ for all $n \in \mathbb{Z}/3\mathbb{Z}$. So, all the simple tensors of this tensor product are equal to 0, and hence $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$ as a module. This shows that the third bullet point above can *really* collapse a tensor product down.

Comment. The subscript R on a tensor product can be important. This is because often one and the same abelian group can be considered as a module over different rings, and you need to declare which one you're thinking of so the tensor product knows which simple tensors to identify when it comes to the third bullet point above. The smaller the ring, the fewer things will be identified, so the tensor product can be much larger. For example, we have said that every $\mathbb{C}[x]$-module is also a $\mathbb{C}$-vector space, but $\dim_{\mathbb{C}} \mathbb{C}[x]/(x^2+x+1) \otimes_{\mathbb{C}[x]} \mathbb{C}[x] = 2$, while $\dim_{\mathbb{C}} \mathbb{C}[x]/(x^2+x+1) \otimes_{\mathbb{C}} \mathbb{C}[x] = \infty$.

On the other hand, lots of times the ring in question is clear from context, so people will suppress the subscript.

Comment. There is a lot of confusion in the world over the correct scope of the word "tensor", because it gets used to mean "an element of any tensor product", or to mean "an element of a really specific kind of tensor product arising in physics". The physicists do not seem to know that's what they mean.

The universal property

Just like everything else, the tensor product also has a universal property, but it's a little bit of a different flavor from the other ones. As I've mentioned, tensor products were invented originally as a kind of "generalized bilinear product". This is true in the sense that given any homomorphism $f\colon M \otimes_R N \to C$, we can define an R-bilinear map $M \times N \to C$ by just sending (m, n) to $f(m \otimes n)$. Bilinearity comes for free because it's built into the definition. What the universal property says is that this relationship goes the other way as well.

Theorem. Let M, N, C be R-modules. If $\phi\colon M \times N \to C$ is an R-bilinear map, then there exists a unique R-module homomorphism $\widetilde{\phi}\colon M \otimes_R N \to C$ such that $\widetilde{\phi}(m \otimes n) = \phi(m, n)$.

Like all of our other universal properties, this gives us a way to define maps out of a tensor product, which otherwise might seem like a somewhat daunting task given that there can be so much weird collapsing that can happen inside a tensor product.

The next few exercises showcase some fundamental things you need to know about tensor products. If you are pretty unfamiliar with tensor products, you should take the results of Exercise 9 to Exercise 15 as axioms, and use them to attempt some problems like Exercise 16, or some qual problems. If you're more comfortable with tensor products, you should deepen your understanding by trying to prove each of these statements using the universal property of tensor products.

Exercise 9. Prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Exercise 10.

(a) Prove that for any R-module M, $M \otimes_R R = M$. If you think of the tensor product as like a multiplication on modules, the module R is like the "unit module" of this multiplication.

(b) Prove that for any multiplicative subset $S \subset R$, $S^{-1}R \otimes_R M \cong S^{-1}M$.

Exercise 11. Prove that $M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$. This makes sense because multiplication distributes over addition.

Exercise 12.

(a) Suppose R is a subring of a larger ring R', so that R' is an R-module. Prove that $R' \otimes_R R[x] \cong R'[x]$.

(b) Consider carefully: what is $R[x] \otimes_R R[x]$?

Fact. A very important fact we will prove in a second, but which is somewhat hard to prove just from what we have seen so far, is that for any R-module M and ideal $I \subset R$, $M \otimes_R (R/I) \cong M/IM$.

Exercise 13. Using the above fact, prove that for any ring R, $R/I \otimes_R R/J \cong R/(I+J)$. Notice that this generalizes Exercise 9.

Exercise 14.

(a) You now have all the information you need to take the tensor product of any two finitely-generated $\mathbb{Z}$-modules. Pick some f.g. $\mathbb{Z}$-modules and tensor those puppies together!

(b) Pick an f.g. $\mathbb{Z}$-module A and tell me:

   (i) What is $A \otimes \mathbb{Z}[\frac{1}{2}]$?

   (ii) What is $A \otimes \mathbb{Z}_{(2)}$?

(iii) What is $A \otimes \mathbb{Q}$?

Exercise 15. (from Vakil) Suppose R_1, R_2, R_3 are rings, and suppose there are ring homomorphisms $R_1 \to R_2$ and $R_1 \to R_3$, so that both R_2, R_3 are R_1-modules.

(a) Let M be an R-module. Prove that $R_2 \otimes_{R_1} M$ is an R_2-modules (in addition to an R_1-module).

(b) Prove that the tensor product $R_2 \otimes_{R_1} R_3$ is a ring.

Exercise 16. A classic example you should be familiar with is computing $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Since $\mathbb{C}$ is 2-dimensional as a real vector space, we know from the Example 1 above that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a 4-dimensional real vector space. From Exercise 15, we know also that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a ring. I only know of three rings that are also 4-dimensional real vector spaces:

- $M_2(\mathbb{R})$,
- $\mathbb{C} \oplus \mathbb{C}$,
- $\mathbb{H}$, Hamilton's algebra of quaternions.

Of these, only $\mathbb{C} \oplus \mathbb{C}$ is commutative, so it would be my guess that this is what we want.

(a) Show or recall that $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$.

(b) Combine several of the above properties to prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

Comment. Perhaps you encountered this already with the above exercises, but often one of the most difficult parts of a problem involving tensor products is verifying that a given tensor is not 0. Altho it is often pretty easy to spot when a tensor *is* 0, it is usually nigh impossible to argue directly from the definition of the tensor product that there is *no* sequence of valid manipulations that can convert a given tensor into the zero tensor. The only piece of technology at your disposal in such situations is the universal property: to prove that your tensor is not 0, use the universal property to construct a map where the image of that tensor is not 0. The first time you do this successfully you will feel more powerful than God.

Tensor products, homomorphisms, and flat modules

For now, fix some R-module N. If we have a homomorphism $f\colon M_1 \to M_2$, then we get an R-bilinear map $M_1 \times N \to M_2 \otimes_R N$ that sends $(m_1, n) \mapsto f(m_1) \otimes n$. Thus, the universal property tells us that we get a homomorphism $\widetilde{f}\colon M_1 \otimes_R N \to M_2 \otimes_R N$. Just like we talked about with localization, if we have a sequence of homomorphisms

$$A \to B \to C,$$

we get a sequence of homomorphisms

$$A \otimes_R N \to B \otimes_R N \to C \otimes_R N,$$

and if our original sequence is exact, we can ask if the sequence we obtain after tensoring is also exact. However, unlike for localization, this need not be the case, as the following classic counterexample demonstrates:

Example 3. Let $R = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, and consider the SES

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Tensoring with N, we get the sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Using our tensor product properties above we can rewrite the terms of this more nicely:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

However, if we examine the map on the left, we started with the muliplication by 2 map, so when we tensor with $\mathbb{Z}/2\mathbb{Z}$ it becomes the multiplication by 0 map! And that is definitely not injective, so the resulting sequence fails to be exact on the left side. (But notice, remains exact in the middle and on the right.)

You can cook up a similar example with $R = \mathbb{C}[x]$, $N = \mathbb{C}[x]/(x)$, and the multiplication by x map. Keep these sequences in mind, they are the best counterexamples to a lot of statements, and frequently appear in one form or another on the qual.

Thus, tensor products can fail to preserve exactness on the left term of a SES, which is the same as saying that if $A \to B$ is injective, it is not necessarily the case that $A \otimes_R N \to B \otimes_R N$ is also injective. On the other hand, it is *always* true that they preserve exactness in the middle and on the right. That is, for any homomorphism $f\colon A \to B$, we have that $(B/\operatorname{im} f) \otimes_R N \cong (B \otimes_R N)/(\operatorname{im} \widetilde{f})$, and in particular, if f is surjective then $\widetilde{f}$ is as well. To capture this, people say that **tensor product is right exact**.

Comment. Somehow this has failed to come up anywhere except in the preceding paragraph. If $f\colon A \to B$ is a homomorphism, the module $B/\operatorname{im} f$ is called the **cokernel** of f, $\operatorname{coker} f$. It is analogous to the kernel, in the sense that the kernel measures how far a map is from being injective, and the cokernel measures how far it is from being surjective. For any homomorphism, we thus get the following four-term exact sequence (the 0s don't count as terms):

$$0 \to \ker f \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0.$$

I could rephrase what I said in the previous paragraph by saying "tensor product preserves cokernels", and people will actually say that! It's not even to be highfalutin, just at some point in your life you realize that is the correct way to say it.

This gives us a quick proof of the fact that I mentioned earlier: $M \otimes_R R/I \cong M/IM$. (This fact is really invaluable!) We start with the SES

$$0 \to I \to R \to R/I \to 0,$$

and we tensor the sequence by M to get a right-exact sequence

$$M \otimes I \to M \otimes R \cong M \to M \otimes R/I \to 0.$$

Now, the map $M \otimes I \to M$ may not be injective, but just by looking at the definition of the map we get from the universal property of the tensor product, we can see that this map sends $m \otimes i \mapsto im$, so we know its image indeed is IM. Thus, exactness tells us that

$$M \otimes R/I \cong M/(\operatorname{im} M \otimes I \to M) = M/IM.$$

Comment. It might seem like $M \otimes_R I$ should literally just be isomorphic to the submodule IM, and the map on the right just be the inclusion, which is injective. This seems supported by our proof that $M \otimes_R R \cong M$, because what we did there was rewrote every simple tensor $m \otimes r$ as $rm \otimes 1$, using the R-bilinearity. So, why can't we just rewrite $m \otimes i$ as $im \otimes 1$? The reason is that the simple tensors in $M \otimes_R I$ *absolutely must* have an element of I on the right side, and except in the trivial situation that $I = (1)$, 1 is not going to be an element of I, so $im \otimes 1$ is just an illegal expression to write down.

We will see in a moment that in a sense, the failure of $M \otimes_R I$ to be isomorphic to IM is the fundamental thing causing the tensor product with M to not be exact on the right.

The next exercise I took from a Michigan qual because I thought it was neat. It demonstrates the behavior I was talking about in the above comment, and ties together several other things we have discussed so far. I have broken it up into a lot of small parts, because there is a subtlety of the shape I mentioned before: you need to prove that the tensor $x \otimes y - y \otimes x$ is not 0, and doing so is not straightforward imo.

Exercise 17. (reworked from a Michigan qual) Let $R = \mathbb{C}[x, y]$, and let $\mathfrak{m} = (x, y)$. Note that $\mathfrak{m}$ is a maximal ideal of R, and that $R/\mathfrak{m} \cong \mathbb{C}$. Since $\mathfrak{m}$ is an ideal of R, it is a torsion-free R-module. The main goal of this problem will be to show that $\mathfrak{m} \otimes_R \mathfrak{m}$ is *not* torsion-free. (!)

(a) Find a generating set for $\mathfrak{m}^2$.

(b) The quotient $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional $\mathbb{C}$-vector space. Find a basis for it.

(c) Find a basis for $\mathfrak{m}/\mathfrak{m}^2 \otimes_R \mathfrak{m}/\mathfrak{m}^2$ as a $\mathbb{C}$-vector space.

(d) Consider now the module $\mathfrak{m} \otimes_R \mathfrak{m}$. The quotient map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ induces a map $\mathfrak{m} \otimes_R \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \otimes_R \mathfrak{m}/\mathfrak{m}^2$. Show that the elements $x \otimes y$ and $y \otimes x$ of $\mathfrak{m} \otimes_R \mathfrak{m}$ get sent to linearly independent vectors under this map. Therefore, $x \otimes y \neq y \otimes x$ as elements of $\mathfrak{m} \otimes_R \mathfrak{m}$.

(e) Verify that $x(x \otimes y) = x(y \otimes x)$, so that $x \otimes y - y \otimes x$ is x-torsion. (It is also y-torsion, for the same reasons.)

(f) There is a short exact sequence of R-modules

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Tensoring with $\mathfrak{m}$ gives the exact sequence

$$\mathfrak{m} \otimes_R \mathfrak{m} \to R \otimes_R \mathfrak{m} \cong \mathfrak{m} \to (R/\mathfrak{m}) \otimes_R \mathfrak{m} \to 0.$$

Prove that the map on the left is not injective.

The failure of tensor products to be exact on the left makes it a special occasion when you find a module N so that tensoring with that module *is* exact. A module N is called **flat** if whenever

$$0 \to A \to B \to C \to 0$$

is a SES, the tensor product sequence

$$0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$$

is also exact. As noted above, we already know that the tensor product sequence will be exact in the middle and on the right, so really the content is that N is flat if tensoring with N *preserves injections.*

Comment. It is also the case that if C is flat, and $0 \to A \to B \to C \to 0$ is an SES, then for any module M,

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is still exact. This was pretty surprising to me when I learned it, it seems like somehow the flatness of C is able to have a mysterious long-range effect on the injectivity of $A \otimes_R M \to B \otimes_R M$. Proving this fact involves some amount of homological algebra, and I don't want to get into that in these notes.

Example 4. We have already met some examples of flat modules.

- R is always a flat module over itself, because if we tensor with R we just get our original modules and our original homomorphism, so of course it preserves injections.
- As a consequence, any free module is flat, since R is flat and tensor products distribute over direct sums.
- For any multiplicative subset $S \subset R$, you showed that $S^{-1}R \otimes_R M \cong S^{-1}M$. Thus, since we already showed that localization is exact, tensoring with $S^{-1}R$ is exact, and so $S^{-1}R$ is a flat R-module.

- As a particular case of the previous point, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, since $\mathbb{Q} = \mathbb{Z}_{(0)}$. Keep this example in mind! We have already shown that $\mathbb{Q}$ is not a finitely-generated $\mathbb{Z}$-module in a previous set of notes, and having a non-finitely-generated flat module on hand is helpful for counterexamples, e.g. coming soon in the section on projective modules.

The definition of flatness seems like it would be impossible to check, because in general it seems like we would have to somehow predict every possible injective homomorphism of modules and verify that all of them stay injective. It would be a pretty sorry state if that was the case, and the following fact tells us that really we only need to check certain homomorphisms that we are already well-acquainted with.

Fact. A module M is flat if and only if for every ideal $I \subset R$, tensoring the injection $I \to R$ by M remains an injection. Equivalently, M is flat if and only if for every ideal I, the map $M \otimes_R I \to IM$ which sends $m \otimes i \mapsto im$ is an isomorphism. (It is automatically a surjection, so it is enough to check that it is injective.)

Exercise 18. Let R be a ring, and M an R-module.

(a) Prove that if M is flat, then M is torsion-free.

(b) Suppose R is a PID. Use the fact above to prove that if M is torsion-free, then M is flat.

(c) What are the finitely-generated flat $\mathbb{Z}$-modules?

The above exercise, and others that have appeared in this section, suggest that the failure of a module to be flat in general has something to do with torsion. In the case of PIDs, you have shown that the torsion determines whether or not the module is flat, but as we saw in Exercise 17, over a non-PID you can have torsion-free modules that fail to be flat. There is a story to be told about trying to measure "how much a module fails to be flat", which is closely related to torsion. I will say no more about it other than to say that this is the reason why Tor is called Tor, if that means something to you.

Flat modules don't appear very often on the qual, which I personally think is a pity, but they did appear in January of this year, in the following problem. I have not yet defined what an R-algebra is, but we've already met them in Exercise 15: an R-algebra is an R-module that also happens to be a commutative ring, so that the ring multiplication commutes with scalar multiplication.

Exercise 19. (January 2024 Problem 4, notation altered) Let R be a commutative ring, and let A and B be commutative R-algebras.

(a) Prove that if B is flat over R, and f is a non zero divisor in A, then $f \otimes 1$ is a non zero divisor in $A \otimes_R B$.

(b) Prove that the flatness condition above is necessary by giving an example where part (a) fails if B is not flat over R.

Comment. I could alternatively have said, a commutative R-algebra A is a commutative ring A with a ring homomorphism $R \to A$, so that A can also be thought of as an R-module. This definition needs to be slightly modified if you want to allow A to be noncommutative, which you should because $M_2(\mathbb{C})$ should be a $\mathbb{C}$-algebra.

Why are they called "flat" modules?

Great question, I'd like to know as well. So would Serre, and he coined the term! (Unless it was maybe Eilenberg or Cartan, but he was the first one to use it in a publication.) If I had to guess, I'd hazard that the motivation might have had to do with the fact that flat modules are torsion-free, so perhaps the phrase "flat" was supposed to capture some idea of being "non-twisted". That's pure speculation, but this much is clear: Serre isolated flatness in a purely algebraic context, as the right condition to preserve certain properties he was interested in. According to Brian Conrad on that mathoverflow post, Serre credits Grothendieck with the insight that in the context of algebraic geometry, flatness is the right condition to ensure that a family of geometric objects varies "continuously" in the right way.

Daniel Erman's answer to that same mathoverflow post discusses this a little more. In that answer, he gives the example (which I'm rephrasing so I don't have to explain what Spec means) of the ring map $k[t] \to k[x, y, t]/(xy - t)$ which sends $t \mapsto t$. Basically, you can understand this ring map geometrically by graphing $xy = t$ on Desmos and watching what happens when you hit play on the t parameter. You can see that the shape of the graph varies nicely continuously, with only a little weirdness at $t = 0$, where the two branches of the hyperbola meet to become the union of the two coordinate axes. But visually, that seems like what they should do. On the other hand, his second example involves the non-flat map $k[t] \to k[x, y, t](t(xy - 1))$. If you were to graph $txy = t$ on Desmos, whenever $t = 0$ you would just get the hyperbola $xy = 1$, but when $t = 0$ you would suddenly get the vacuous equation $0 = 0$, whose solutions are all the points of the plane. This kind of bizarre behavior is the sort of thing flatness prevents.

If you're interested, this old course webpage has links to the original papers in which Serre uses the term "flat" (French "plat") in both the original French and English translations.

Tensor products over noncommutative rings

I have avoided talking about tensor products over noncommutative rings, because I didn't want to clog up the exposition fussing about left versus right modules, but it is important to know that you can take tensor products over noncommutative rings as well. Here are the key things that can be different for noncommutative rings:

1. In order to form the tensor product $M \otimes_R N$, M must be a right R-module and N must be a left R-module.

2. The relation $(rm) \otimes n = m \otimes (rn) = r(m \otimes n)$ for commutative rings is replaced by the relation $mr \otimes n = m \otimes rn$.

3. Because we can no longer "pull the r out", the resulting tensor product will not necessarily be an R-module, only an abelian group. It will only be an R-module if one of M or N also has an additional R-module structure on the other side. A module over a noncommutative ring that has a structure as both a left and a right R-module is called an R-**bimodule**.

4. The universal property will no longer reference R-bilinear maps, but rather will reference what are called R-balanced maps. An R-**balanced map** $\phi\colon M \times N \to P$ is a function that is
   - additive in each component, i.e. $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$, and similarly on the right; and,
   - allows elements of R to "pass across the arguments": $\phi(mr, n) = \phi(m, rn)$.

The most important tensor product properties are still true for tensor products over noncommutative rings. In particular:

- $M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$, and similarly on the left.
- If I is a left ideal of R, then we can form the quotient R/I as an abelian group (but not as a ring), and it will be a left R-module. In this case, we have that $M \otimes_R R/I \cong M/MI$. There is a similar property for right ideals.

Exercise 20. (January 2021 Problem 2, modified) Let $R = M_2(\mathbb{C})$, the non-commutative ring of 2×2 matrices over $\mathbb{C}$.

(a) Give examples of a simple left R-module M and a simple right R-module N.

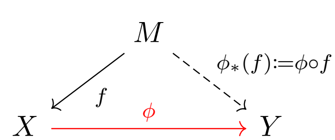
(b) Can you find another simple left R-module M' that is not isomorphic to M? (Either find one or explain why there is none such.)

(c) Compute $\dim_{\mathbb{C}}(N \otimes_R M)$. (**NB** You can do this by working directly with the tensors.)

Projective modules, and maybe a hair about injective modules

We return to a bit of possibly more familiar territory, to discuss a few more properties of the modules $\mathrm{Hom}_R(M, N)$.

Our previous experience with localizations and tensor products suggests that it would be interesting to find out how the modules $\mathrm{Hom}_R(M, N)$ interact with sequences. For now, we fix a module M. If we have a homomorphism $\phi\colon X \to Y$, we get a map $\phi_*\colon \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y)$ by sending $f \mapsto \phi \circ f$. (You may hear this being called a "pushforward".) I think it's helpful to have a picture in your head of this:



What I'm saying above is that given any red arrow ϕ, we can transform any $f\colon M \to X$ into a function $\phi_*(f)\colon M \to Y$ by "sliding the tip of f along ϕ". It sort of resembles the definition of vector addition. Now, given a SES

$$0 \to A \to B \to C \to 0$$

we get a sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0.$$

Is this sequence exact?

No, and we can turn to our old counterexample

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

If we apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to each term of the sequence we get the sequence

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

This new sequence is not exact because the it's not exact at the $\mathbb{Z}/2\mathbb{Z}$ term on the right, since the map $0 \to \mathbb{Z}/2\mathbb{Z}$ is not surjective.

Exercise 21. We can modify this sequence a bit to make it a little less trivial-looking. Consider the sequence

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

(a) Apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to each of the terms of this sequence, and write down the sequence you get at the end.

(b) Confirm that this is again an example where the resulting sequence is not exact at the right term, but is exact in the middle and on the left.

This behavior holds in general: it is always true that taking $\mathrm{Hom}_R(M, -)$ preserves exactness in the middle and on the right, but it may not preserve exactness on the left, i.e. may not preserve surjections. We say that **Hom is left exact**. Saying that the sequence stays exact on the middle and right is the same as saying that for any homomorphism $\phi\colon A \to B$, we have that $\mathrm{Hom}_R(M, \ker\phi) = \ker\phi_*$.

Just like with tensor products, the fact that taking Hom can fail to be exact makes it interesting whenever we find a module M so that taking Hom *is* exact. A module M is called **projective** if whenever $0 \to A \to B \to C \to 0$ is exact, so too is

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0.$$

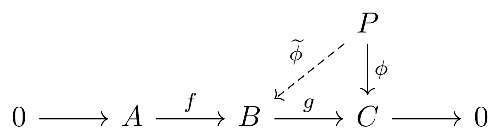
Exercise 22. Prove that a free module is projective. That is, let F be free, and let $\phi \colon B \to C$ be surjective, and prove that $\phi_* \colon \mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(F, C)$ is also surjective. (You may assume that F is finite rank, but it is not necessary.)

It turns out we can be a lot more specific with characterizing projective modules than we can with flat modules.

Exercise 23. Prove that the following are equivalent.

(i) The module P is projective.

(ii) has the following **lifting property**: whenever $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a SES, and $\phi \colon P \to C$ is any map, there exists a map $\widetilde{\phi} \colon P \to B$ so that $\phi = g \circ \widetilde{\phi}$. In a diagram:



(iii) Any exact sequence $0 \to A \to B \to P \to 0$ splits.

(iv) The module P is a direct summand of a free module.

Hints:

(i) ⇒ (ii) Apply $\mathrm{Hom}_R(P, -)$ to the sequence. Notice that we know that $\mathrm{id}_P \in \mathrm{Hom}_R(P, P)$.

(iii) ⇒ (iv) You may use that any module is a quotient of a free module. (We already discussed this in the case of f.g. modules.)

(iv) ⇒ (i) If F is a free module and $F = P \oplus Q$, show that any homomorphism $P \to C$ can be extended to a homomorphism $F \to C$. Then, use that free modules are projective.

Example 5.

- I think I didn't quite state this outright, but if R is a PID, then any submodule of a free module is free (no restriction to finitely-generated modules). As a consequence, a module over a PID is projective iff it is free.
- Consider $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module. By CRT, we have that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, so $\mathbb{Z}/2\mathbb{Z}$ is a direct summand of a free module, hence is projective. However, any free f.g. $\mathbb{Z}/6\mathbb{Z}$-module has order a power of 6, so $\mathbb{Z}/2\mathbb{Z}$ cannot be a free module.
- For a noncommutative example, consider the ring $M_2(\mathbb{C})$ and the simple left module $V = \mathbb{C}^2$. We showed at the top of this document that $M_2(\mathbb{C}) \cong V \oplus V$, so V is a projective module, but similar to the previous example, V cannot be free because any free module has dimension divisible by 4 as a complex vector space.

- A long time ago we talked about the ring $R = \mathbb{Z}[\sqrt{-5}]$, which had the curious property that 6 has two factorizations into irreducibles: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. It turns out that the ideal $I = (2, 1+\sqrt{-5}) \subset R$ is not a principal ideal, and therefore it is not a free R-module. However, it *is* a projective R-module. In fact,

$$v_1 = \begin{pmatrix} 2 \\ 1-\sqrt{-5} \end{pmatrix},\ v_2 = \begin{pmatrix} 1+\sqrt{-5} \\ 2 \end{pmatrix}$$

  forms a basis for $I \oplus I$, so $I \oplus I \cong R^{\oplus 2}$.

Exercise 24. (January 2022 Problem 3 (b)) Let M be a projective module over R and $S \subset R$ a multiplicative set. Prove that $S^{-1}M$ is a projective $R[S^{-1}]$-module.

Exercise 25. (January 2021 Problem 3) Two questions about projective modules.

(a) Show that a projective module over an integral domain is torsion free.

(b) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated modules over a commutative, Noetherian ring. Prove the following OR give a counterexample:

 (i) If M_1 and M_2 are projective, then so is M_3.

 (ii) If M_2 and M_3 are projective, then so is M_1.

Exercise 26. (Projective modules are flat)

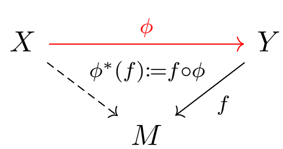
(a) Prove that a projective module is flat.

(b) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, so we need not have an implication in the other direction.

A fun fact that I didn't learn until recently is that for f.g. modules over *any* ring, M is flat iff M is projective. The way you would prove this is by showing that both conditions are equivalent to M being **locally free**, which means that for every maximal ideal $\mathfrak{m} \subset R$, $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module.

Homs the other way

We have talked a lot about taking Homs like $\mathrm{Hom}_R(M, -)$, but you could equally well talk about taking Homs like $\mathrm{Hom}_R(-, M)$. This presents an interesting new behavior: given a homomorphism $\phi\colon X \to Y$, we get a homomorphism going *the other direction* $\phi^*\colon \mathrm{Hom}_R(Y, M) \to \mathrm{Hom}_R(X, M)$ (also called a "pullback"). In diagram form:



This time, we slide the tail of $f\colon Y \to M$ backwards along the red arrow ϕ to get a function $\phi^*(f)\colon X \to M$. Notice again the similarity to vector addition. To distinguish between the two kinds of Hom, people will call $\mathrm{Hom}_R(M, -)$ **covariant Hom** (because the homomorphisms go the same way), and $\mathrm{Hom}_R(-, M)$ **contravariant Hom** (because the homomorphisms swap).

Like before one can ask what happens to exact sequences

$$0 \to A \to B \to C \to 0.$$

They flip around, and it turns out they don't remain exact, but

$$\mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$$

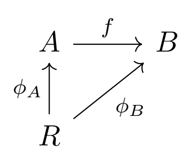
will still be exact. It's a little bit of a puzzle to decide what sidedness of exactness to call this, but the common consensus is that we should call contravariant Hom **left exact**, just like covariant Hom. (It's the left side of the original diagram that it preserves, it just mirror images it along the way.)

Just like with covariant Hom, there's a special name for the modules that make this exact. They are called **injective modules**. They're a lot less nice that projective modules, but there are good reasons to learn about them that I won't get into here. I will simply note that the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective, and in general injective modules have a feeling of having "highly dividable elements".

Algebras over a ring

I just wanted to mention briefly the definition of an "algebra" over a commutative ring R. We've already discussed that an **R-algebra** is an R-module A that is also a ring (not necessarily commutative), with the property that scalar multiplication by elements of R commutes with the ring multiplication in A. We can also define an R-algebra as a (not necessarily commutative) ring A together with a ring homomorphism $\phi\colon R \to A$, sometimes called the **structure homomorphism**. In order to make sure that scalar multiplication commutes with ring multiplication, i.e./ $a_1(\phi(r)a_2) = \phi(r)a_1a_2$, we need for the image of ϕ to consist entirely of elements that commute with every $a \in A$. The subset of A consisting of elements that commute with every other element is called the **center** of A, and is denoted $Z(A)$.

An **R-algebra homomorphism** between two R-algebras A, B is a ring homomorphism $A \to B$ that is also an R-module homomorphism. Equivalently, if A, B have structure homomorphisms ϕ_A, ϕ_B, an R-algebra homomorphism between them is a ring map $f\colon A \to B$ so that $\phi_B = f \circ \phi_A$. That is, so that the following diagram commutes.



You already know several examples of algebras.

Example 6.

- Given a commutative ring R, any quotient ring is an R-algebra.
- Any polynomial ring $R[x_1, \ldots, x_n]$ is an R-algebra, as is any quotient of a polynomial ring.
- The ring of $n \times n$ matrices with entries in R, $M_n(R)$, is an R-algebra.
- More generally, for any R-module M, $\mathrm{End}_R(M)$ is an R-algebra.
- Given a field K, any field extension L/K is a K-algebra.
- The field of real numbers $\mathbb{R}$ has a very interesting 4-dimensional algebra, Hamilton's algebra of quaternions $\mathbb{H}$. There is a more general notion of something called a "quaternion algebra" over any field K, inspired by Hamilton's quaternions.

Comment. A minor warning, some people in the world will use the word "algebra" to mean something more general than what it means here. For example, some people want to let the octonions $\mathbb{O}$ be an $\mathbb{R}$-algebra, and would prefer to call the algebras we've defined "associative algebras".