Group Theory 2
Group actions
by Ivan Aidun

These are notes interspersed with exercises. The purpose of these notes is to be more fleshed out than Evan Dummit's old notes, but shorter and more focused than a textbook treatment. Halfway between Dummit and Dummit and Foote, so to speak. I hope these are helpful to you!

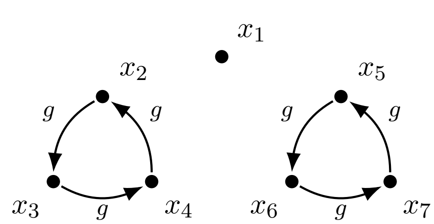
Group Actions

Group actions are the most important thing about groups. Their importance is witnessed by the fact that the groups D_n, S_n, GL_n, and SL_n are essentially defined as "the group that acts on the set X" for different sets X. This is part of the power of abstract algebra, we can take a group whose elements are really actions on some other set, and treat it as if those actions were small enough to hold in your hand.

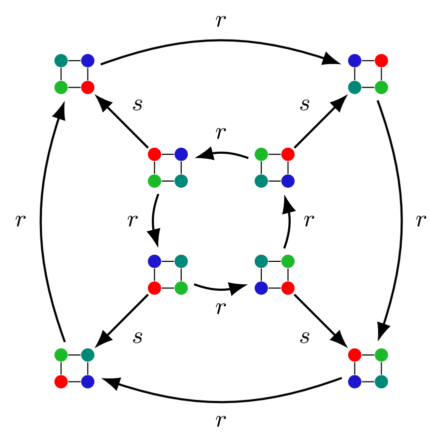
Formally, a group action is like a "scalar multiplication" by elements of a group on a set X. Because groups can be noncommutative, we need to specify whether we want our groups to act on the left or the right. Most of the time I will write actions on the left, but you can't get away from right actions sometimes. Then, for a group G to act on a set X on the left, often written $G \circlearrowright X$, we require that for each $x \in X$ and $g \in G$, the element $g \cdot x \in X$, such that $e \cdot x = x$ and $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$.

Equivalently, a group action is the same thing as for each $g \in G$, specifying a function $\phi_g \colon X \to X$, where the function $\phi_g(x) = g \cdot x$. The two latter requirements above can be rephrased as $\phi_e = \mathrm{id}_X$, and $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$. Notice that because $g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = e \cdot x = x$, the functions ϕ_g are in fact all bijections $X \to X$. Denoting the group of bijections $X \xrightarrow{\sim} X$ by S_X (by analogy to the symmetric group S_n), we can further rephrase a group action $G \circlearrowright X$ as a group homomorphism $G \to S_X$. Sometimes it is more convenient to view a group action in this way, and sometimes it is more convenient to view it more as a scalar multiplication.

If the set X is finite, you can visualize a group action by drawing a graph. For example, let C_6 be the (multiplicatively written) cyclic group of order 6 generated by g. Here is a possible action of C_6 on a set of size 7.



Below is a more complicated graph of D_4 acting on a square. Here, the generator r is rotation counterclockwise by 45°, and s is reflection about the diagonal going from upper right to lower left.



Here is some commonly-used terminology relating to group actions:

- Sometimes a set X with a group action $G \circlearrowright X$ is called a G-**set**.

- Given an element $x \in X$, the set of elements of X that can be reached by applying some $g \in G$ is called the **orbit of** x, which I will denote $G \cdot x$, but some people prefer $\mathrm{Orb}_G(x)$. So, in the first example $G \cdot x_1 = \{x_1\}$ while $G \cdot x_2 = \{x_2, x_3, x_4\}$. In the second example, there is only one orbit, and we can get from any position of the square to any other position by applying an appropriate group element. An action where there is only one orbit is called **transitive**.

- An element $x \in X$ is a **fixed point** of the action if $G \cdot x = \{x\}$. The set of fixed points is denoted X^G. In the first example, $X^G = \{x_1\}$, while in the second example $X^G = \emptyset$. The standard notation for the subset of X fixed by a specific $g \in G$ is X^g, which I find really easy to get confused with X^G, so I'm going to write $\mathrm{Fix}(g)$ instead.

- Given an element $x \in X$, the set of $g \in G$ so that $g \cdot x = x$ is called the **stabilizer of** x. The stabilizer of a given $x \in X$ is always a subgroup of G. I will denote the stabilizer of x by $\mathrm{Stab}(x)$, but some people denote G_x. Again, I just find it confusing to try to keep track of the differences between $G \cdot x, X^G, X^g$, and G_x, and only the first two are common enough and sensible enough for me to keep straight. So, in the first example, $\mathrm{Stab}(x_1) = C_6$ (corresponding to being a fixed point) and $\mathrm{Stab}(x_2) = \{e, g^3\}$. In the second example, every stabilizer is equal to $\{e\}$. An action where every stabilizer is trivial is called **free**.

- A less-commonly used term, the **kernel** of an action is the set of $g \in G$ that fix every $x \in X$. When we think about an action as a homomorphism $G \to S_X$, the kernel of the action coincides with the kernel of the homomorphism. As a consequence, the kernel of an action is always a normal subgroup of G. In the first example, the kernel is $\{e, g^3\}$, while in the second example the kernel is $\{e\}$.

- An action that has trivial kernel is called a **faithful action**. So, the second example is faithful, while the first is not. To put it another way, an action is faithful iff $g \cdot x = h \cdot x$ for all $x \in X$

implies that $g = h$. The idea is that a faithful action "faithfully encodes" all the different elements of G as different functions on X, without collapsing any distinctions. A free action is always faithful, but not necessarily the other way around. **Exercise 1.** Give an example.

- Sometimes an action that is both transitive and free will be called **simply transitive**. The action of D_4 above is simply transitive.

If group actions are the most important thing about groups, then the Orbit-Stabilizer Theorem is the most important thing about group actions.

Theorem (Orbit-Stabilizer)**.** Suppose G is a group, X is a set, and $G \circlearrowright X$. For each $x \in X$, the map

$$\begin{aligned} G &\to (G \cdot x) \\ g &\mapsto g \cdot x \end{aligned}$$

is surjective and $|\mathrm{Stab}(x)|$-to-one. In particular, if G is finite, we obtain that

$$\frac{\#G}{\#\,\mathrm{Stab}(x)} = \#(G \cdot x).$$

Exercise 2. (Lagrange's theorem) Let G be a finite group, $H < G$, and denote by $[G : H]$ the number of cosets of H in G. Use the Orbit-Stabilizer Theorem to prove that $\#G = [G : H] \cdot (\#H)$, and in particular that both $\#H$ and $[G : H]$ divide $\#G$.

Exercise 3. (Class equation)

(a) Let G be a finite group, and let $G \circlearrowright X$. Let $x_1, \ldots, x_n$ be representatives of each orbit in X. Prove that

$$\#X = \sum_{i=1}^{n} \#(G \cdot x_i).$$

(b) Let $g_1, \ldots, g_k \in G$ be representatives for the conjugacy classes of size greater than 1. Prove that

$$\#G = \#Z(G) + \sum_{i=1}^{k} \#[g_i].$$

(c) Use the Orbit-Stabilizer Theorem to prove that $\#[g_i] \mid \#G$.

Exercise 4. (Cauchy's theorem) Let G be a finite group, and let $p \mid \#G$ be a prime number. Consider the set $X \subset G^p$ consisting of all the tuples $(g_1, \ldots, g_p)$ satisfying $\prod_{i=1}^{p} g_i = e$.

(a) Let the cyclic group $C_p = \langle \sigma \rangle$ act on X by cyclic permutation: $\sigma \cdot (g_1, \ldots, g_p) = (g_2, \ldots, g_p, g_1)$. Prove that $\sigma \cdot (g_1, \ldots, g_p) \in X$, so this action is well-defined.

(b) Prove that $\#X = (\#G)^{p-1}$.

(c) Prove that the number of tuples in X fixed by σ is divisible by p. Prove also that this number is at least 1.

(d) Prove that there is an element $g \in G$ of exact order p.

I'm not going to state or prove the Sylow theorems, because they don't appear on the qual anymore, but the proof of the is similar to the above proof of Cauchy's theorem, except you have to be both more careful and more clever. As Jordan Ellenberg once said, the Sylow theorems are hard to prove because you need to have not one but *two* good ideas.

Exercise 5. (January 2024 Problem 3) Let G be a group acting transitively on a set X. Let $H \subset G$ be a normal subgroup. Then H naturally acts on X, but the action need not be transitive. Prove that all orbits of H on X have the same cardinality. (You may assume G and X are finite if you like, but this is not necessary!)

Exercise 6. (August 2022 Problem 2) Let $\mathbb{F}_p$ be the finite field with p elements; here p is a prime number. Let $V = \mathbb{F}_p^2$, and recall that $G = \mathrm{GL}_2(\mathbb{F}_p)$ is the group of invertible linear transformations on V. G acts on V in the usual way (by multiplication).

(a) Describe the orbits of this action.

(b) Describe the stabilizer in G of the vector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V.$$

(c) Consider now the action of G on $V \times V$ (acting independently on each of the two vectors). How many orbits does this action have?

(d) What is the cardinality of G? Remember to justify your answer.

Exercise 7. (August 2023 Problem 5, edited) By S_n we mean the symmetric group on n elements.

(a) Let G be a subgroup of S_n. Suppose the action of G on $\{1, \ldots, n\}$ is transitive. (**NB** such a subgroup is called a "transitive" subgroup.) Prove that for any $x, y \in \{1, \ldots, n\}$, the stabilizer of x in G and the stabilizer of y in G are conjugate subgroups of G.

(b) Prove that if G is a transitive abelian subgroup of S_n, then $|G| = n$.

(c) Give an example of an integer n and a subgroup G of S_n such that G is transitive and abelian but not cyclic.

(d) For which integers n does S_n contain a transitive abelian subgroup G that is not cyclic? (Justify your answer.)